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Novel exploration of nonlinear waves of a generalized (2+1)-dimensional extended Boussinesq equation: solitons, breathers and periodic background waves

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Abstract: In this analytical investigation, we explore various solutions for a generalized and extended Boussinesq (eBO) equation in (2+1)-dimensions. It applies a problem-solving approach known as the generalized exponential rational function technique, which first transforms the equation into a simplified ordinary differential equation under a wave transformation. It considers the trial function as a rational function involving exponential functions for the simplified equation. With these considerations, the desired analytic solutions, which include solitons, breathers, kink-solitons, and periodic background waves, are explored for the investigated equation. Under arbitrary choices for the constants in the rational function, it analyzes different families with several computed cases to obtain the solutions for a rational function, which provides the general solution of the studied eBO equation using back substitution. We graphically explore the obtained solutions in the form of different solitons, breathers, and periodic waves, with arbitrary choices for the involved constant coefficients. It discusses the work's significant importance and the dynamic behavior of the obtained solution. The investigated eBO equation is important in nonlinear sciences and has many physical applications. It provides insights into the multi-dimensional waves and solitons interactions in applied mathematics and physics and many other fields such as optics, fluid dynamics, geophysics, and plasma physics.

1 Introduction

Nonlinear partial differential equations (NLPDEs) are used as a tool to describe complex physical phenomena having effect of nonlinearity and dispersion. KdV and schrödinger equations are the fundamental NLPDEs to study the waves in shallow and deep water respectively. Both equation generates the solitary wave structures that balance the nonlinearity and dispersion for the studied PDEs. Solitary waves are imortant to study as they preserve their shape and size throughout the wave propagation. These waves are the funcdamental nonlinear wave solutions that preserve the shape even after the collision to the other waves. Obtaining the analytic or exact solutions for such PDEs are important to gain the knowledge of the behaviors of the waves and their interactions in several fields such as fluid dynamics, oceanography, optics, plasma physics, and other nonlinear sciences. Due to the lack of general solutions of NLPDEs, there are different approaches to obtain the analytic and exact solutions including including the simplified Hirota's technique [1, 2], Darboux transformation [3, 4], Bäcklund transformation [5, 6], Symbolic bilinear technique [7], Bilinear Neural Network Method [8, 9], Direct symbolic approach [10–14], Hirota's bilinear approach [15–18], and other methodologies [19].

This research investigates the (2+1)-dimensional extended Boussinesq (eBO) equation [20] as

$$u_{xt} + \alpha(u_{xxxx} + 6u_x u_{xx}) + \beta u_{tt} + a u_{xx} + b u_{xy} = 0, \tag{1}$$

having u as the wave/amplitude function of independent variables x, y and t with constant coefficients α, β, a, and b, related to the model's physical parameters that describe the propagation of nonlinear waves in higher-dimensional system. Wazwaz et al. (2024) [20] proposed and analyzed this equation, which is a sophisticated generalization of the classic shallow-water Boussinesq model. It captures richer dynamics in (3+1)-dimensional fluid environments. This extended version of the Boussinesq equation incorporates additional higher-order nonlinear and dispersive terms. The additional terms enable it to describe complex wave behaviors such as solitons, shock waves, periodic structures, and localized lumps. Through Painlevé analysis, the authors demonstrated that this model is Painlevé-integrable. Moreover, Hirota bilinear approach was applied to obtain analytic solutions, including multi-soliton, kink, periodic, and lump (localized) structures. Physically, this means that the extended Boussinesq equation can model the interaction of nonlinear wave forms on shallow water surfaces in multiple dimensions. The model's ability to produce both coherent soliton trains and complex localized features makes it especially valuable. It studies phenomena ranging from internal waves in stratified fluids to nonlinear pulse propagation in optical or plasma systems.
The classic shallow-water Boussinesq model is an introductory equation in fluid dynamics that describes the behavior of long surface waves in shallow water under the influence of gravity. Mathematically, the classic Boussinesq equation [21] takes a form such as

$$u_{tt} - c^2 u_{xx} - a(u^2)_{xx} + b u_{xxxx} = 0, \tag{2}$$

where $u(x, t)$ is the surface elevation, c is the wave speed in linear theory, and the terms involving a and b represent nonlinear and dispersive effects, respectively. It captures the delicate balance between nonlinearity and dispersion, which tends to steepen wavefronts and spread out waves, respectively. It makes the equation suitable for modeling weakly nonlinear, weakly dispersive waves that can be observed in coastal and near-shore environments. A pivotal advantage of the Boussinesq model is its ability to simulate bidirectional wave propagation, which is not possible in simpler models, such as the Korteweg–de Vries (KdV) equation that describes waves traveling in one direction. Due to this reason, the Boussinesq model has been widely applied in various fields such as tsunami modeling, wave-current interactions, and coastal engineering. However, the classical form of the equation becomes less accurate when applied to strongly nonlinear or multi-dimensional scenarios.

The manuscript is structured as: The following section 2 provides an overview of the applied generalized exponential rational function technique, including its steps. These steps include the formation of ordinary differential equations through wave transformation. The third section 3 obtains the analytic solution for the investigated eBO equation for different families, allowing for an arbitrary choice of real parameters. It also displays the dynamics of the obtained solutions with various choices of constants. In section 4, we analyze the results with analytic observations and discuss the several forms of the wave structures. Ultimately, we conclude our research work.

2 Overview of problem-solving technique: GERFM

To find the analytic solutions of the (2+1)-dimensional generalized extended Boussinesq (eBO) equation, we utilize a well known generalized exponential rational function method (GERFM) [22–24]. This method can be explored generally into steps as

- Let us consider a (2+1)-dimensional nonlinear partial differential equations (PDE)

$$P(\upsilon, \upsilon_x, \upsilon_y, \upsilon_t, \upsilon_{xx}, \upsilon_{xt}, \upsilon_{yt}, \upsilon_{tt} \cdots) = 0, \tag{3}$$

  and apply the traveling wave transformation $\upsilon(x, y, t) = G(\phi)$ where $\phi = sx + ky + jt + \lambda$, then the studied Nonlinear PDE (3) converts into an ordinary differential equation (ODE)

$$Q(G, G^{'}, G^{"}, G^{'''} \ldots) = 0. \tag{4}$$

- We suppose the solution of the equation (4) as

$$G(\phi) = J_0 + \sum_{i=1}^{N} J_i M(\phi)^i + \sum_{i=1}^{N} K_i M(\phi)^{-i} \tag{5}$$

  where N is the balancing constant obtained by using homogeneous balance principle, and $M(\phi)$ is a rational function

$$M(\phi) = \frac{\omega_1 e^{\eta_1 \phi} + \omega_2 e^{\eta_2 \phi}}{\omega_3 e^{\eta_3 \phi} + \omega_4 e^{\eta_4 \phi}}, \tag{6}$$

  with arbitrary constants $\omega_i, \eta_i, (1 \leq i \leq 4)$ and constant coefficients J_0, J_i and $K_i (1 \leq i \leq N)$.

- On substituting the equation (5) with (6) into the equation (4), collecting all the possible powers of $\{e^{\phi}\}$, and equating their coefficients C_j for the integer j to zero, forms an algebraic system $C_j = 0$.

- At the end, after solving the system of equations, we will substitute the obtained values into the equations (5) and (6) that establishes the analytic solutions of the ODE (4). Further doing back substitution, we create the analytic solution for the investigated eBO equation (1).

3 Analytic solutions of eBO equation

The study in the work aims to find out the different types of analytic solutions for the studied nonlinear eBO equation (1) through various forms as soliton, kink-type soliton, lump-chain, breather and periodic background waves. Now utilizing the GERFM to the studied equation as per discussed in above section, we have following process as

Considering the wave transformation

$$u(x,y,t) = G(\phi); \quad \phi = sx + ky + jt + \lambda, \tag{7}$$

where s, k and λ are arbitrary constants. On substituting the equationo (7) into (1), we get an transformed equation in the form of an ordinary differential equation (ODE) as

$$jsG''[\phi] + bksG''[\phi] + as^2G''[\phi] + j^2\beta G''[\phi] + \alpha(6s^3G'[\phi]G''[\phi] + s^4G^{(4)}[\phi]) = 0. \tag{8}$$

With the help of homogeneous balancing principle, we balance the terms $G^{(4)}$ and $G'G''$ of the equation (8), we deduce $N+4 = (N+1)+(N+2)$, which implies that $N = 1$. Hence, from equation (5), we get the trial solution as

$$G(\phi) = J_0 + J_1M(\phi) + \frac{K_1}{M(\phi)}, \tag{9}$$

where $M(\phi)$ is rational function as in equation (6). Next, we substitute the equation (9) into (8) and follow the steps of the GERFM. To obtain the different solutions, we consider different families for different values of the constants in the rational function (6).

Family 1: For $[\omega_1, \omega_2, \omega_3, \omega_4] = [-6, 7, 1, 1]$ and $[\eta_1, \eta_2, \eta_3, \eta_4] = [1, 0, 1, 0]$, then the equation (6) becomes

$$M(\phi) = \frac{7 - 6e^{\phi}}{1 + e^{\phi}}. \tag{10}$$

On substituting equation (10) into (9), we get

$$G(\phi) = \frac{K_1(1+e^{\phi})}{7-6e^{\phi}} + \frac{J_1(7-6e^{\phi})}{1+e^{\phi}} + J_0 \tag{11}$$

On putting the equation (11) with (10) into the equation (8), and collecting all the possible powers of $Y_j = \{\mathrm{e}^{\phi}\}^{\mathrm{j}}$ for some integer j, forms an algebraic system $Y_j = 0$ for all j. On solving the obtained system we get values as

Case 1.1:

$$J_0 \neq 0,\ K_1 = -\frac{84s}{13},\ J_1 = 0,\ k = \frac{-js - as^2 - s^4\alpha - j^2\beta}{bs}.$$

Substituting the values of the above constants into equation (11), we get a solution for (8) as

$$G(\phi) = -\frac{84(1+e^{\phi})s}{13(7-6e^{\phi})} + J_0$$

Consequently, an analytic solution of (1) is obtained as

$$u(x,y,t) = J_0 - \frac{84(1+e^{jt+sx+ky+\lambda})s}{13(7-6e^{jt+sx+ky+\lambda})} \tag{12}$$

Case 1.2:

$$J_0 \neq 0,\ J_1 = -\frac{2s}{13},\ K_1 = 0,\ k = \frac{-js - as^2 - s^4\alpha - j^2\beta}{bs}.$$

Substitute the values of the above constants into equation (11), then equation (8) gives:

$$G(\phi) = -\frac{2(7-6e^{\phi})s}{13(1+e^{\phi})} + J_0$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x,y,t) = J_0 - \frac{2(7-6e^{jt+sx+ky+\lambda})s}{13(1+e^{jt+sx+ky+\lambda})} \tag{13}$$

Case 1.3:

$$J_0 \neq 0,\ K_1 = \frac{-84s}{13},\ J_1 = 0,\ j = \frac{-s+\sqrt{s^2-4bks\beta-4as^2\beta-4s^4\alpha\beta}}{2\beta}.$$

Substitute the values of the above constants into equation (11), then equation (8) gives:

$$G(\phi) = -\frac{84(1+e^{\phi})s}{13(7-6e^{\phi})} + J_0.$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x,y,t) = J_0 - \frac{84(1+e^{jt+sx+ky+\lambda})s}{13(7-6e^{jt+sx+ky+\lambda})} \tag{14}$$

Case 1.4:

$$J_0 \neq 0,\ K_1 = \frac{-84s}{13},\ J_1 = 0,\ j = \frac{-s-\sqrt{s^2-4bks\beta-4as^2\beta-4s^4\alpha\beta}}{2\beta}.$$

Substitute the values of the above constants into equation (11), then equation (8) gives:

$$G(\phi) = -\frac{84(1+e^{\phi})s}{13(7-6e^{\phi})} + J_0.$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x,y,t) = J_0 - \frac{84(1+e^{jt+sx+ky+\lambda})s}{13(7-6e^{jt+sx+ky+\lambda})} \tag{15}$$

Family 2: For $[\omega_1, \omega_2, \omega_3, \omega_4] = [-1,-1,1,-1]$ and $[\eta_1, \eta_2, \eta_3, \eta_4] = [1,1,1,0]$, then equation (6) becomes,

$$M(\phi) = -\frac{2e^{\phi}}{-1+e^{\phi}} \tag{16}$$

Next, we substitute equation (16) into (9) and we get:

$$G(\phi) = -\frac{1}{2}e^{-\phi}(-1+e^{\phi})K_1 - \frac{2e^{\phi}J_1}{-1+e^{\phi}} + J_0 \tag{17}$$

Case 2.1:

$$J_0 \neq 0,\ J_1 = -s,\ K_1 = 0,\ j = \frac{-s-\sqrt{s^2-4bks\beta-4as^2\beta-4s^4\alpha\beta}}{2\beta}.$$

Substitute the values of the above constants into equation (17), then equation (8) gives:

$$G(\phi) = \frac{2e^{\phi}s}{-1+e^{\phi}} + J_0.$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x,y,t) = J_0 + \frac{2e^{jt+sx+ky+\lambda}s}{-1+e^{jt+sx+ky+\lambda}} \tag{18}$$

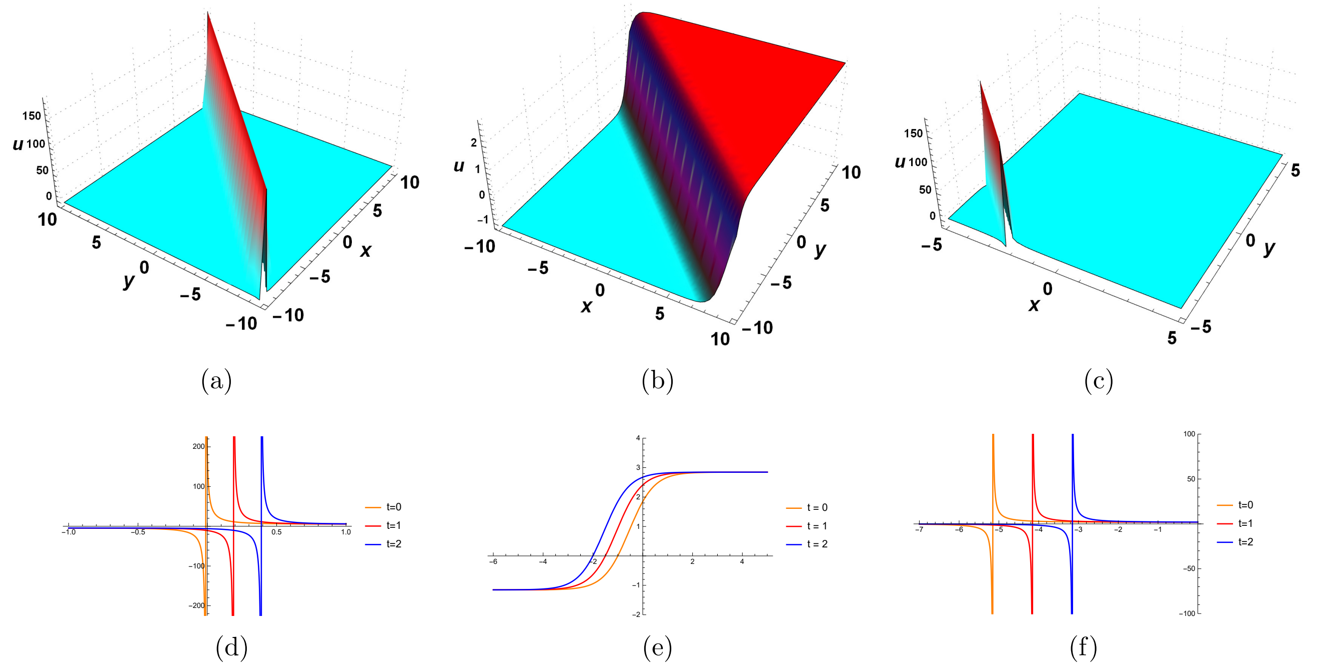


Figure 1: Graphics for the solutions (12), (13), and (15) as in (a) (b) and (c) respectively, with their 2D plots in (d), (e), and (f), having parameters : (a) $t = 0$, $j = 1$, $s = -5$, $k = 5$, $\lambda = .1$, $J_0 = 1$; (b) $t = 0$, $j = 1$, $s = 2$, $k = 2$, $\lambda = 1.1$, $J_0 = 1$; (c) $t = 0$, $j = 1$, $s = -1$, $k = -1$, $\lambda = -7$, $J_0 = 1$.

Case 2.2:

$$J_0 \neq 0,\ J_1 = -s,\ K_1 = 0,\ j = \frac{-s + \sqrt{s^2 - 4bks\beta - 4as^2\beta - 4s^4\alpha\beta}}{2\beta}$$

. Substitute the values of the above constants into equation (17), then equation (8) gives:

$$G(\phi) = \frac{2e^{\phi}s}{-1 + e^{\phi}} + J_0.$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x, y, t) = J_0 + \frac{2e^{jt+sx+ky+\lambda}s}{-1 + e^{jt+sx+ky+\lambda}} \tag{19}$$

Case 2.3:

$$J_0 \neq 0,\ J_1 = -s,\ K_1 = 0,\ k = \frac{-js - as^2 - s^4\alpha - j^2\beta}{b\, s}.$$

Substitute the values of the above constants into equation (17), then equation (8) gives:

$$G(\phi) = \frac{2e^{\phi}s}{-1 + e^{\phi}} + J_0.$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x, y, t) = J_0 + \frac{2e^{jt+sx+ky+\lambda}s}{-1 + e^{jt+sx+ky+\lambda}} \tag{20}$$

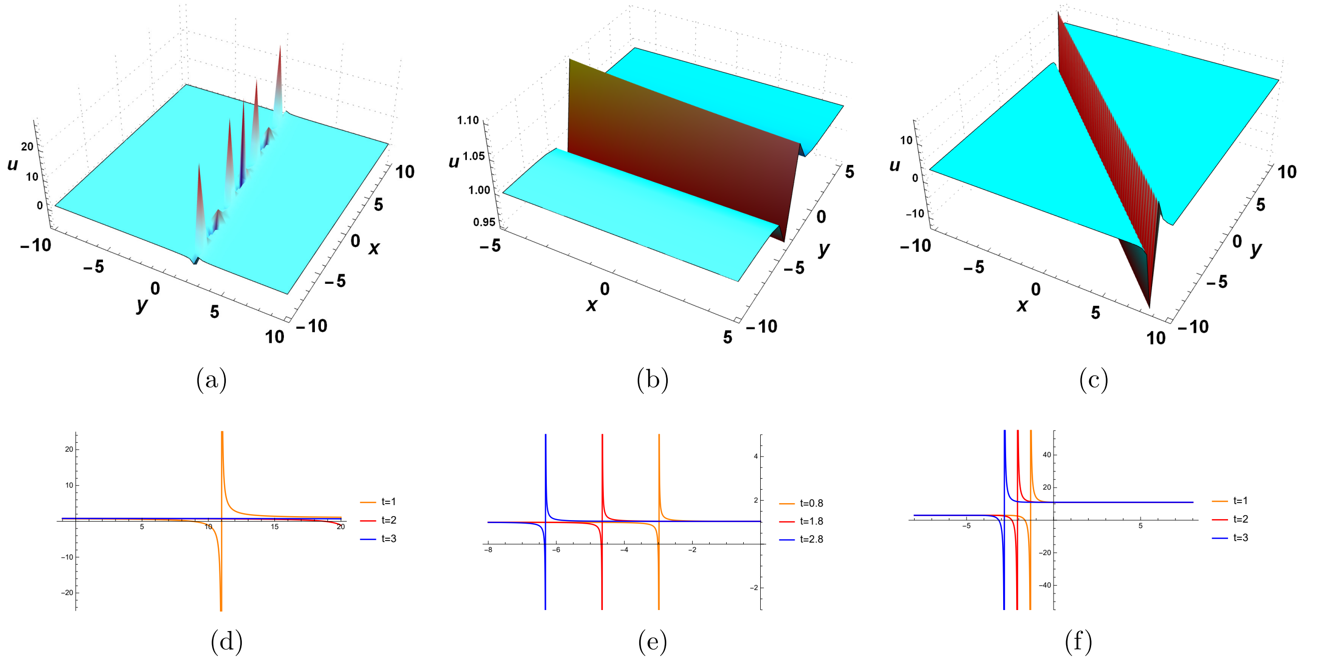


Figure 2: Graphics for the solutions (18), (19), and (20) as in (a) (b) and (c) respectively, with their 2D plots in (d), (e), and (f), having parameters : (a) $t = 1$, $j = 1$, $s = -.1$, $k = -.8$, $\lambda = .1$, $J_0 = 1$; ; (b) $t = .8$, $j = 2$, $s = .02$, $k = 1.2$, $\lambda = 2$, $J_0 = 1$; ; (c) $t = 1$, $j = 3$, $s = 4$, $k = 4$, $\lambda = 2.3$, $J_0 = 3$.

Family 3: For $[\omega_1, \omega_2, \omega_3, \omega_4] = [2, 2, 2, -2]$ and $[\eta_1, \eta_2, \eta_3, \eta_4] = [-2, -2, -2, 2]$, then the equation (6) becomes,

$$M(\phi) = \frac{4e^{-2\phi}}{2e^{-2\phi} - 2e^{2\phi}} \tag{21}$$

Next, we substitute equation (21) into (9) and we get:

$$G(\phi) = \frac{1}{4}e^{2\phi}(2e^{-2\phi} - 2e^{2\phi})K_1 + \frac{4e^{-2\phi}}{2e^{-2\phi} - 2e^{2\phi}}J_1 + J_0 \tag{22}$$

Case 3.1:

$$J_0 \neq 0,\ J_1 = -4s,\ K_1 = 0,\ \beta = -\frac{s(j + bk + as + 16s^3\alpha)}{j^2}.$$

Substitute the values of the above constants into equation (22), then equation (8) gives:

$$G(\phi) = -\frac{16e^{-2\phi}s}{2e^{-2\phi} - 2e^{2\phi}} + J_0$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x, y, t) = J_0 - \frac{16e^{-2(jt+sx+ky+\lambda)}s}{2e^{-2(jt+sx+ky+\lambda)} - 2e^{2(jt+sx+ky+\lambda)}} \tag{23}$$

Case 3.2:

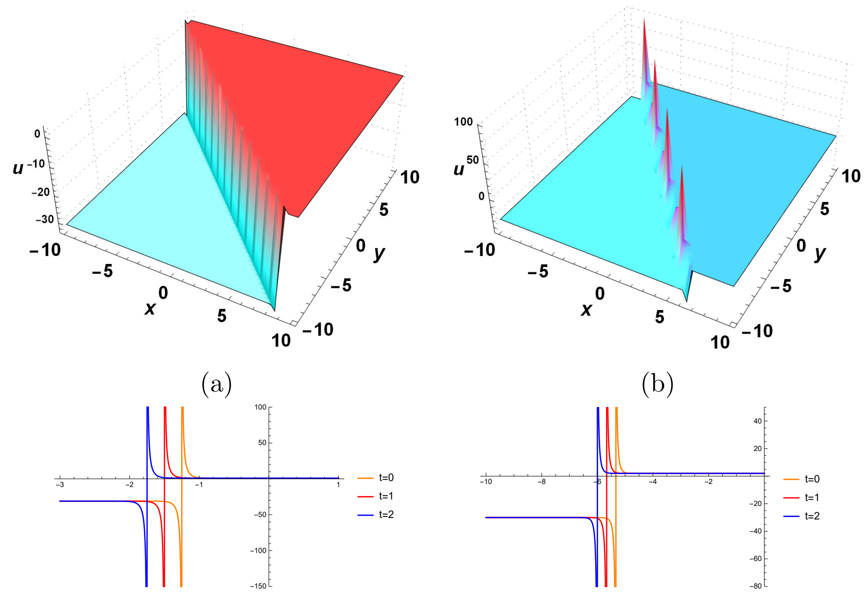
$$J_0 \neq 0,\ J_1 = -4s,\ K_1 = 0,\ \alpha = \frac{-js - bks - as^2 - j^2\beta}{16s^4}.$$

Substitute the values of the above constants into equation (22), then equation (8) gives:

$$G(\phi) = -\frac{16e^{-2\phi}s}{2e^{-2\phi} - 2e^{2\phi}} + J_0$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x, y, t) = J_0 - \frac{16e^{-2(jt+sx+ky+\lambda)}s}{2e^{-2(jt+sx+ky+\lambda)} - 2e^{2(jt+sx+ky+\lambda)}} \tag{24}$$



(a) (b) (c) (d)

Figure 3: Graphics for the solutions (23), and (24) as in (a) and (b) respectively, with their 2D plots in (c) and (d), having parameters : (a) $t = 0,\ j = 1,\ s = 4,\ k = 4,\ \lambda = 5,\ J_0 = 1$; ; (b) $t = 0,\ j = 1,\ s = 4,\ k = 3,\ \lambda = 4,\ J_0 = 2$.

Family 4: For $[\omega_1, \omega_2, \omega_3, \omega_4] = [i, -i, -i, 1]$ and $[\eta_1, \eta_2, \eta_3, \eta_4] = [-i, i, -i, i]$, then equation (6) becomes,

$$M(\phi) = \frac{ie^{-i\phi} - ie^{i\phi}}{-ie^{-i\phi} + e^{i\phi}} \tag{25}$$

Next, we substitute equation (25) into (9) and we get:

$$G(\phi) = \frac{(-ie^{-i\phi} + e^{i\phi})K_1}{ie^{-i\phi} - ie^{i\phi}} + \frac{(ie^{-i\phi} - ie^{i\phi})J_1}{-ie^{-i\phi} + e^{i\phi}} + J_0 \tag{26}$$

Case 4.1:

$$J_0 \neq 0,\ K_1 = (2+2i)s,\ J_1 = 0,\ j = \frac{-s - \sqrt{s^2 - 4bks\beta - 4as^2\beta + 16s^4\alpha\beta}}{2\beta}.$$

Substitute the values of the above constants into equation (26), then equation (8) gives:

$$G(\phi) = \frac{(2+2i)(-ie^{-i\phi} + e^{i\phi})s}{ie^{-i\phi} - ie^{i\phi}} + J_0$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x,y,t) = \frac{(2+2i)(-ie^{-i(jt+sx+ky+\lambda)} + e^{i(jt+sx+ky+\lambda)})s}{ie^{-i(jt+sx+ky+\lambda)} - ie^{i(jt+sx+ky+\lambda)}} + J_0 \tag{27}$$

Case 4.2:

$$J_0 \neq 0,\ J_1 = (-2+2i)s,\ K_1 = 0,\ j = \frac{-s - \sqrt{s^2 - 4bks\beta - 4as^2\beta + 16s^4\alpha\beta}}{2\beta}.$$

Substitute the values of the above constants into equation (26), then equation (8) gives:

$$G(\phi) = -\frac{(2-2i)(ie^{-i\phi} - ie^{i\phi})s}{-ie^{-i\phi} + e^{i\phi}} + J_0$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x,y,t) = -\frac{(2-2i)(ie^{-i(jt+sx+ky+\lambda)} - ie^{i(jt+sx+ky+\lambda)})s}{-ie^{-i(jt+sx+ky+\lambda)} + e^{i(jt+sx+ky+\lambda)}} + J_0 \tag{28}$$

Case 4.3:

$$J_0 \neq 0,\ J_1 = 0,\ K_1 = (2+2i)s,\ k = \frac{-js - as^2 + 4s^4\alpha - j^2\beta}{b\,s}.$$

Substitute the values of the above constants into equation (26), then equation (8) gives:

$$G(\phi) = \frac{(2+2i)(-ie^{-i\phi} + e^{i\phi})s}{ie^{-i\phi} - ie^{i\phi}} + J_0$$

Consequently, an exact soliton solution of (1) is obtained, as follows:

$$u(x,y,t) = \frac{(2+2i)(-ie^{-i(jt+sx+ky+\lambda)} + e^{i(jt+sx+ky+\lambda)})s}{ie^{-i(jt+sx+ky+\lambda)} - ie^{i(jt+sx+ky+\lambda)}} + J_0 \tag{29}$$

Case 4.4:

$$J_0 \neq 0,\ J_1 = (-2+2i)s,\ K_1 = 0,\ k = \frac{-js - as^2 + 4s^4\alpha - j^2\beta}{b\,s}.$$

Substitute the values of the above constants into equation (26), then equation (8) gives:

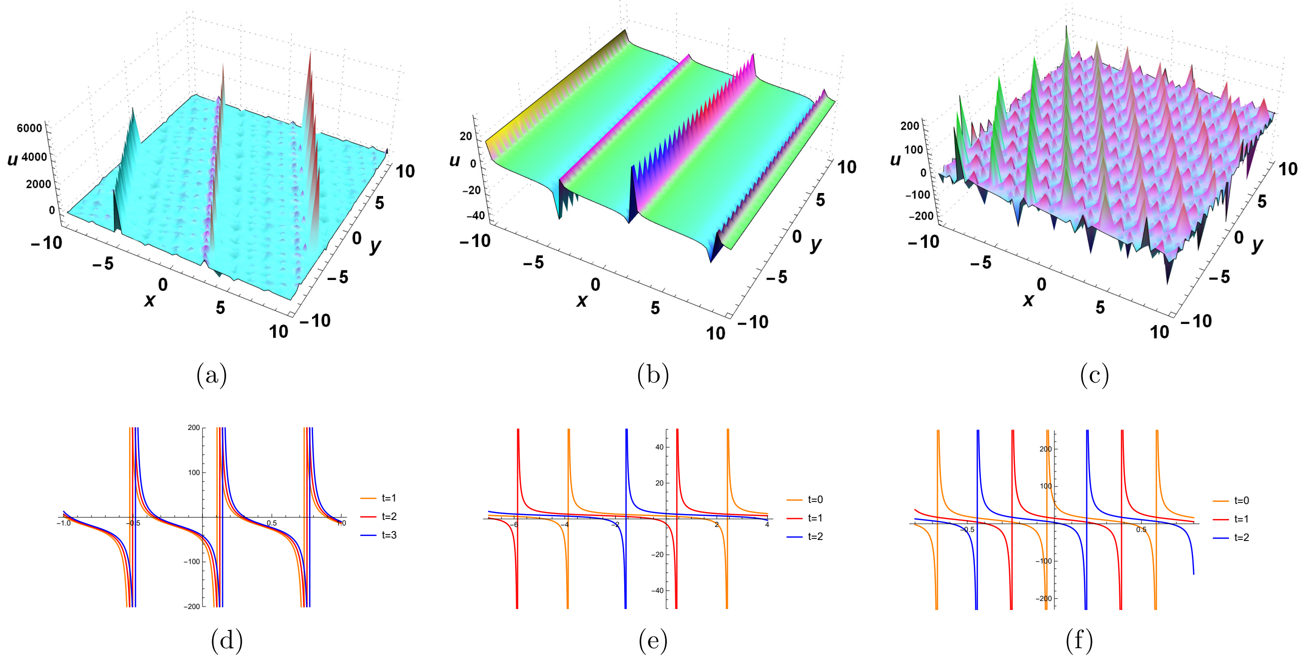
$$G(\phi) = -\frac{(2-2i)(ie^{-i\phi} - ie^{i\phi})s}{-ie^{-i\phi} + e^{i\phi}} + J_0$$

Consequently, an exact soliton solution of (1) is obtained, as

$$u(x,y,t) = -\frac{(2-2i)(ie^{-i(jt+sx+ky+\lambda)} - ie^{i(jt+sx+ky+\lambda)})s}{-ie^{-i(jt+sx+ky+\lambda)} + e^{i(jt+sx+ky+\lambda)}} + J_0 \tag{30}$$

4 Results and analysis

This work analyzed the different analytical solutions by considering the different famalies for the trial function in the utilized GERF method. This approach consider the trial function as a rational functions with arbitrary parameters. For different values of arbitrary parameters, we get the different trail functions and hence different analytical solutions. The dynamical analysis of these obtained solutions are analyzed with the symbolic software *Mathematica*, for appropriate choices of the arbitrary parameters. The two-dimentional graphics are plotted for different time t values.



(a) (b) (c)

(d) (e) (f)

Figure 4: Graphics for the solutions (27), (28), and (30) as in (a) (b) and (c) respectively, with their 2D plots in (d), (e), and (f), having parameters : (a) $t = 1$, $j = -.1$, $s = 10$, $k = 5$, $\lambda = 9$, $J_0 = 1$; (b) $t = 0$, $j = 1$, $s = 0.5$, $k = .001$, $\lambda = 9$, $J_0 = 1$; (c) $t = 0$, $j = 1$, $s = 5$, $k = 3$, $\lambda = 1$, $J_0 = 1$.

The startiing choices of arbitrary parameters in trail function gives the opportunities to go with the different analytical solutions that showcase the different wave solutions such as solitons, kinks, peridoic soliton, perodic background waves and other wave structures. The studied solutions represents the different type of wave structures depending on the arbitrary choice of parameters and show the waves behavior due to the nonlinearity and dispersion of the investigated equation.
The analysis for the drawn figures is followed as

- Figure 1 shows the dynamical analysis for the different solution cases of Family 1. Plot 1 and 3 depict the single solitons moving in the positive direction of x-axis, and Plot 2 depicts the kink-soliton moving in the nagetive direction of the x-axis. 3D plots are shown in the xy-coyrdinates, and 2D plots are shown with respect to the different time values as $t = 0, 1,$ and 2.

- In Figure 2, we analyze the dynamical behavior for the different solution cases of Family 2. Plot 2 and 3 depict the bright-dark single solitons moving in the nagetive direction of x-axis, and Plot 1 depicts the periodic-soliton moving in the positive direction of the x-axis. 3D plots are shown in the xy-coyrdinates, and 2D plots are shown with respect to the different time values as (d) $t = 1, 2, 3$, (e) $t = 0.8, 1.8, 2.8$, and (f) $t = 1, 2, 3$.

- Figure 3 depicts the graphical analysis for the different solution cases of Family 3. Plot 1 shows the kink-soliton moving in the nagetive direction of x-axis, and Plot 2 shows a periodic-soliton moving in the nagetive direction of x-axis. 3D plots are shown in the xy-coyrdinates, and 2D plots are shown with respect to the different time values as $t = 0, 1,$ and 2.

- In Figure 4, we analyze the dynamical analysis for the different solution cases of Family 4. Plot 1 and 3 shows the solitons with periodic background moving in the positive and nagetive direction of x-axis, respectively; and Plot 2 shows a periodic-soliton moving in the nagetive direction of x-axis. 3D plots are shown in the xy-coyrdinates, and 2D plots are shown with respect to the different time values as (d) $t = 1, 2, 3$, (e) and (f) $t = 0, 1, 2$.

5 Conclusions

This research investigated the analytic solutions of a (2+1)-dimensional eBO equation with the generalized exponential rational function method. An ordinary differential equation was obtained, having a wave transformation for the studied partial differential equation. Utilizing the trial function for analytic solutions, we obtained several exponential solutions with different families of arbitrary choices for the constant parameters. We generated solutions in the form of rational and exponential functions with several arbitrary parameters. Dynamical analysis for the different obtained solutions was performed using the symbolic software *Mathematica*. These dynamics revealed the formation of various wave structures, including solitons, periodic solitons or breathers, kink solitons, and solitons with a periodic background. The wave structures depicted the significance of nonlinearity and dispersion present in the studied eBO equation with the different values of the suitable parameters. Generated solitons balance the nonlinearity and dispersion as in KdV or Schrödinger equation, which are fundamental studies for the solitary waves. The investigated eBO equation plays a crucial role in nonlinear sciences, particularly in the study of solitary waves, which have significant physical implications in various scientific fields, including nonlinear sciences, applied mathematics, and physics.

Declarations

Ethics approval and consent to participate

Not applicable.

Competing interests

There is no conflict of interest, according to the authors.

Authors' contributions

Each author made an equal contribution to the final draft of the work. The authors would have consented and approved the final work.

Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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