Symbolic computation of bilinear equation for the KdV-type partial differential equations

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Abstract

In this work, we focus on blinear equation, a crucial tool in study of nonlinear partial differential equations(NLPDEs).Using this technique, we also compare our results with already known methods like Hirota's technique.To convert a nonlinear partial differential equation (PDE) into bilinear form, the Hirota method is used.Numerous disciplines, including nonlinear dynamics, visual science, mathematical physics such as plasma physics, thermo-mechanics, optical science, and engineering sciences, employ the Hirota technique, a strong and precise mathematical tool, to find soliton solutions of nonlinear PDEs.These solitons can change depending on different values, which helps us to understand them better. We use a method called the Cole-Hopf transformation to make the equation easier to solve. In order to write a class of nonlinear PDEs in bilinear form, we offer a novel organized mathematical method. This approach is easy to utilize in programs like Mathematica and Maple because to its simplicity. The solutions are derived in simpler forms by performing transformations based on dependent variables and using tried-and-true mathematical techniques. These results show that the effectiveness of the computational algorithm.

1 Introduction

Physics and applied mathematics have a major area that deals with non linear partial differential equations. These equations include unknown functions and their partial changes(called derivatives). These equations are used to describe many physical systems, such as fluid flow, plasma physics, and ocean waves. They have also been used to solve difficult problems like the Poincaré and Calabi conjectures. There is no general method to solve all non-linear PDEs, so each equation is studied separately. Different techniques are used, such as:Darboux transformation (DT) [1,2], Bäcklund transformation [3,4], Hirota bilinear technique [5–7,9], Simplified Hirota method [10–13],Lie symmetry analysis [14,15], Inverse scattering method [16,17], Pfaffian technique [18, 19]. These and other methods help us study non-linear PDEs. Among these, the Hirota method (especially the direct method) [5] is considered the best for finding multi-soliton solutions of integrable non-linear PDEs.
Changing a nonlinear partial differential equation (PDE) into a bilinear form is often a difficult and time-consuming task, even if we already know how to change the variables. That's why it is helpful to create an algorithm (a step-by-step method) that can do this work automatically. With the use of computers, programs like Mathematica and Maple can be quite helpful in performing these computations. Many researchers have studied nonlinear PDEs because they are important in describing real-world problems.
The Hirota method gives exact solutions like solitons, breathers, rogue waves, lump solutions, and others. Studying solitons is non-linear systems is an interesting and important research area. It helps us how a single wave behave in different physical systems. Solitons are employed in non-linear research to investigate localized and steady waves. For various significant nonlinear equations, such as KdV, modified KdV, and nonlinear Schrödinger equations, Hirota and Hietarinta established a straightforward and easy way to determine the accurate N-soliton solutions (a type of wave solution) [20, 21]. They used a special rule called Hirota's 3-soliton condition, which helps in working with nonlinear equations. Using this rule, they searched for easier forms (called bilinear forms) of these complex equations, which makes solving them much easier.
In Section 2, an effective step-by-step algorithm is introduced to find the bilinear equation for nonlinear partial differential equations (PDEs).
In Section 3, several examples of well-known nonlinear PDEs are shown, such as the KdV equation [5], Boussinesq equation [22], SK equation [23], Caundrey-Dodd-Gibbon(CDG) equation [24], KP equation [25], generalized BKP equation [26] and more. These examples come from areas such as nonlinear dynamics, mathematical physic, plasma physics, and other scientific fields. All the work is done using the computer software Mathematica. Understanding how solitons are formed, how they move and interact in non linear systems gives deep knowledge about non-linearity wave spreading, and other important factors. Studying solitons also improves or basic and non linear events. This have real world importance, helps with technological progress and useful in fields like ocean engineering, plasma physics, telecommunication and many nonlinear sciences.

2 General Description For Bilinear Equation

In this section we discusses about the procedure for getting bilinear equation.
Step 1: Firstly we consider a non-linear PDE as

$$Q(u, ux1, ux2, ...) = 0$$

that contains $u = u(x_1, x_2, ..., x_n, t)$ and its partial derivatives with respect to the independent variables x1,x2,...,xn and t.

Step 2: Considering the phase variable η_i depending on the given nonlinear PDE as

$$\eta_i = a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3 + ... + d_it,$$

where $a_{Ni}; 1 \leq N \leq n$ are the constants, and that d_i is the dispersion.To show how the algorithm can be applied to various cases, which are covered in the sections that follow, we now employed the conventional relation for the phase variable. Nevertheless, the particular form of the nonlinear PDE may influence this decision.

Step 3: Finding dispersion relation, a relation between frequencies and wave numbers.

Step 4: Finding Cole-Hopf transformation $u(x,t) = R(lnf)_{nx}$; n=1,2,3... , for the given nonlinear PDE, n can be found by balancing the nonlinear terms with the highest-order derivative.

Step 5: Considering the function $f(x,t) = 1 + e^{\eta_1}$ and substituting in given non linear PDE.

Step 6: Calculate the value of constant R.

Step 7: Finding a bilinear equation in f for non linear PDEs as

$$F(f, fx1, fx2, ...) = 0,$$

which includes f and its partial derivatives taken with respect to the independent variables x1, x2,...,xn and t.

3 Application of Bilinear equation from Non linear

3.1 (1+1)-Dimensional Equations

3.1.1 Korteweg-de varies(KdV) equation

The non-linear KdV equation [5] is given

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

Here, u is the dependent variable that shows the wave's height (amplitude), x stands for the position (space), and t stands for time. Equation (1) mainly focuses on soliton solutions, which are special single-wave solutions that keep their shape and speed as they move. Let us consider a phase variable η in the KdV equation (1) as

$$\eta_i = p_i x - q_i t, \tag{2}$$

where p_i, $i = 1, 2, \ldots$ as constant parameters and q_i as dispersion coefficient
Putting

$$u = e^{\eta_i}$$

then

$$u_t = -q_i e^{\eta i},$$

$$u_{xxx} = p_i^3 e^{\eta i},$$

putting in linear term of eq. (1) $u_t + u_{xxx} = 0$, we get

$$-q_i e^{\eta i} + p_i^3 e^{\eta i} = 0,$$

$$e^{\eta i}(p_i^3 - q_i) = 0,$$

we get dispersion relation

$$q_i = p_i^3.$$

Considering the transformation

$$u(x,t) = R(lnf)_{xx}, \tag{3}$$

and putting it with

$$f(x,t) = 1 + e^{\eta_1}$$

in Eq. (1) On solving,

$$\left(R\left(\frac{p^2 e^{-p^3 t+px}}{1+e^{-p^3 t+px}} - \frac{p^2\left(e^{-p^3 t+px}\right)^2}{\left(1+e^{-p^3 t+px}\right)^2}\right)\right)_t$$
$$+6R\left(\frac{p^2 e^{-p^3 t+px}}{1+e^{-p^3 t+px}} - \frac{p^2\left(e^{-p^3 t+px}\right)^2}{\left(1+e^{-p^3 t+px}\right)^2}\right)\left(R\left(\frac{p^2 e^{-p^3 t+px}}{1+e^{-p^3 t+px}} - \frac{p^2\left(e^{-p^3 t+px}\right)^2}{\left(1+e^{-p^3 t+px}\right)^2}\right)\right)_x$$
$$+\left(R\left(\frac{p^2 e^{-p^3 t+px}}{1+e^{-p^3 t+px}} - \frac{p^2\left(e^{-p^3 t+px}\right)^2}{\left(1+e^{-p^3 t+px}\right)^2}\right)\right)_{x,x,x} = 0$$

$$\Rightarrow \quad -\frac{6Rp^5 e^{-2p(p^2t-x)}\left(e^{-p(p^2t-x)}-1\right)(R-2)}{\left(1+e^{-p(p^2t-x)}\right)^5} = 0,$$

$$\Rightarrow \quad R = 2.$$

So, the logarithmic transformation becomes

$$u(x,t) = 2(lnf)_{xx}, . \tag{4}$$

We can write it in another form

$$u = w_{xx} \quad where \quad w = 2(lnf)_{xx}. \tag{5}$$

Now from (5),we have

$$u_t = w_{xxt}, \quad u_x = w_{xxx} \quad \text{and} \quad \mathrm{u_{xxx} = w_{xxxxx}},$$

putting the above expression into eq.(1),we get

$$w_{xxt} + 6w_{xx}w_{xxx} + w_{xxxxx} = 0.$$

On integrating w.r.t x

$$w_{xt} + 6\int w_{xx}w_{xxx}\partial x + w_{xxxx} = 0, \tag{6}$$

$$I = 6\int w_{xx}w_{xxx}\partial x = \frac{1}{2}\int 2w_{xx}w_{xxx}\partial x = \frac{1}{2}w_{xx}^2,$$

substituting the value of I in eq.(6), we get

$$w_{xt} + 3w_{xx}^2 + w_{xxxx} = 0, \tag{7}$$

then we calculate,

$$\begin{aligned} w_x &= \frac{2f_x}{f}, \\ w_{xt} &= \frac{2(f_{xt}f - f_x f_t)}{f^2}, \\ w_{xx} &= \frac{2(f_{xx}f - f_x^2)}{f^2}, \\ w_{xxx} &= \frac{2(f_{xxx}f^2 - 6f_{xx}f_x f + 4f_x^3)}{f^3}, \\ w_{xxxx} &= \frac{2(f_{xxxx}f^3 - 8f_{xxx}f_x f^2 + 12f_{xx}^2 f^2 + 24 f_{xx} f_x^2 f - 12 f_x^4)}{f^4}, \end{aligned}$$

putting the all above values in equation(7),we get a billinear equation in f as

$$ff_{xt} - f_x f_t + 3{f^2}_{xx} - 4f_x f + ff_{xxxx}. \tag{8}$$

3.1.2 Boussinesq equation

We have the Boussinesq equation [27] as

$$u_{2t} - u_{2x} - 6u_x^2 - 6uu_{2x} - u_{4x} = 0. \tag{9}$$

We suppose the phase variable $\eta_i = p_i x - q_i t$. On putting $u = e^{\eta_i}$ in eq.(9), then we get dispersion relation

$$\begin{aligned} u_{2t} &= q_i^2 e^{\eta i}, \\ u_{4x} &= p_i^4 e^{\eta i}, \\ u_{2x} &= p_i^2 e^{\eta i}, \end{aligned}$$

putting in linear term of eq.(9)$u_{2t} - u_{2x} - u_{4x} = 0$, we get

$$q_i^2 e^{\eta i} - p_i^2 e^{\eta i} - p_i^4 e^{\eta i} = 0,$$

$$e^{\eta i}(p_i^4 + p_i^2) = q_i^2 e^{\eta i}.$$

So, we get dispersion relation as

$$d_i = \sqrt{p_i^2 + p_i^4}.$$

Considering the Cole-Hopf transformation

$$u(x,t) = R(lnf)_{xx}, \tag{10}$$

for the given nonlinear PDE.

$$R\left(\frac{p^2 e^{px-\sqrt{p^4+p^2}\,t}}{1+e^{px-\sqrt{p^4+p^2}\,t}} - \frac{p^2\left(e^{px-\sqrt{p^4+p^2}\,t}\right)^2}{\left(1+e^{px-\sqrt{p^4+p^2}\,t}\right)^2}\right)_{t,t} - \left(R\left(\frac{p^2 e^{px-\sqrt{p^4+p^2}\,t}}{1+e^{px-\sqrt{p^4+p^2}\,t}}\right.\right.$$

$$-\frac{p^2\left(e^{px-\sqrt{p^4+p^2}\,t}\right)^2}{\left(1+e^{px-\sqrt{p^4+p^2}\,t}\right)^2}\Bigg)\Bigg)_{x,x}-6\left(R\left(\frac{p^2e^{px-\sqrt{p^4+p^2}\,t}}{1+e^{px-\sqrt{p^4+p^2}\,t}}-\frac{p^2\left(e^{px-\sqrt{p^4+p^2}\,t}\right)^2}{\left(1+e^{px-\sqrt{p^4+p^2}\,t}\right)^2}\right)^2\right)_x$$

$$-6R(\left(\frac{p^2e^{px-\sqrt{p^4+p^2}\,t}}{1+e^{px-\sqrt{p^4+p^2}\,t}}-\frac{p^2\left(e^{px-\sqrt{p^4+p^2}\,t}\right)^2}{\left(1+e^{px-\sqrt{p^4+p^2}\,t}\right)^2}\right)\left(R\left(\frac{p^2e^{px-\sqrt{p^4+p^2}\,t}}{1+e^{px-\sqrt{p^4+p^2}\,t}}-\frac{p^2\left(e^{px-\sqrt{p^4+p^2}\,t}\right)^2}{\left(1+e^{px-\sqrt{p^4+p^2}\,t}\right)^2}\right)\right)_{x,x}$$

$$-\left(R\left(\frac{p^2e^{px-\sqrt{p^4+p^2}\,t}}{1+e^{px-\sqrt{p^4+p^2}\,t}}-\frac{p^2\left(e^{px-\sqrt{p^4+p^2}\,t}\right)^2}{\left(1+e^{px-\sqrt{p^4+p^2}\,t}\right)^2}\right)\right)_{x,x,x,x}=0$$

$$\Rightarrow\quad \frac{-12(R-2)Re^{2px-2\sqrt{p^2(p^2+1)}\,t}\left(-3e^{px-\sqrt{p^2(p^2+1)}\,t}+e^{2px-2\sqrt{p^2(p^2+1)}\,t}+1\right)p^6}{\left(1+e^{px-\sqrt{p^2(p^2+1)}\,t}\right)^6}=0$$

$$\Rightarrow\quad R=2.$$

So, the logarithmic transformation becomes

$$u(x,t)=2(lnf)_{xx}. \tag{11}$$

We can write it in another form

$$u=w_{xx}\quad where\quad w=2(lnf)_{xx}, \tag{12}$$

now from (12),we have

$$u_{tt}=w_{xxxt},\quad u_{xx}=w_{xxxx}\quad and\quad u_{xxx}=w_{xxxxxx},$$

putting the above expression into eq.(9),we get

$$w_{xxtt}+w_{xxxx}-3(w_{xx})^2_{xx}-w_{xxxxxx}=0$$

on integrating w.r.t x

$$w_{xtt}+w_{xxx}-3(w_{xx})^2_{x}-w_{xxxxx}=0, \tag{13}$$

on again integrating w.r.t x

$$w_{tt}+w_{xx}-3(w_{xx})^2-w_{xxx}=0, \tag{14}$$

then we calculate

$$w_x=\frac{2f_x}{f},$$
$$w_{xx}=\frac{2(f_{xx}f-f_x^2)}{f^2},$$
$$w_{xxx}=\frac{2(f_{xxx}f^2-3f_{xx}f_xf+2f_x^3)}{f^3},$$
$$w_{tt}=\frac{2(f_{tt}f-f_t^2)}{f^2},$$

putting the all above values in equation(9),we get a billinear equation in f as

$$-f_t^2+ff_{tt}+f_x^2-ff_{xx}-3f_{xx}^2+4f_xf_{xxx}-ff_{xxxx}=0. \tag{15}$$

3.1.3 Caundrey-Dodd-Gibbon (CDG) equation

Taking CDG equation [28] as

$$u_t + u_{5x} + 30uu_{3x} + 30u_x u_{2x} + 180u^2 u_x = 0. \tag{16}$$

We suppose the phase variable $\eta_i = p_i x - q_i t$. On putting $u = e^{\eta_i}$ in eq.(16), then

$$u_t = -q_i e^{\eta i},$$

$$u_{5x} = p_i^5 e^{\eta i},$$

putting in linear term of eq.(16)$u_t + u_{5x} = 0$,

$$-q_i e^{\eta i} + p_i^5 e^{\eta i} = 0,$$

$$e^{\eta i}(p_i^5) = q_i e^{\eta i}.$$

Then we get dispersion relation as

$$q_i = p_i^5$$

Considering the Cole-Hopf transformation

$$u(x,t) = R(lnf)_{xx}, \tag{17}$$

for the given nonlinear PDE.

$$R\left(\left(\frac{p^2 e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{(1+e^{-p^5t+px})^2}\right)\right)_t + \left(R\left(\frac{p^2 e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{(1+e^{-p^5t+px})^2}\right)\right)_{x,x,x,x,x}$$
$$+30R\left(\frac{p^2 e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{(1+e^{-p^5t+px})^2}\right)\left(R\left(\frac{p^2 e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{(1+e^{-p^5t+px})^2}\right)\right)_{x,x,x}$$
$$+30\left(R\left(\frac{p^2 e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{(1+e^{-p^5t+px})^2}\right)\right)_x\left(R\left(\frac{p^2 e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{(1+e^{-p^5t+px})^2}\right)\right)_{x,x}$$
$$+180R^2\left(\frac{p^2 e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{(1+e^{-p^5t+px})^2}\right)^2\left(R\left(\frac{p^2 e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{(1+e^{-p^5t+px})^2}\right)\right)_x = 0$$

$$\Rightarrow \frac{180\, e^{-2p(p^4t-x)}\, p^7\, R(R-1)\left(\left(R-\frac{5}{3}\right)e^{-2p(p^4t-x)} + \left(-R+\frac{5}{3}\right)e^{-p(p^4t-x)} + \frac{e^{-3p(p^4t-x)}}{3} - \frac{1}{3}\right)}{\left(1+e^{-p(p^4t-x)}\right)^7 = 0}$$

$$\Rightarrow R = 1$$

We get the value of R is 1, by putting the function $f(x,t) = 1 + e^{\eta_1}$ in equation(16) .
So, the logarithmic transformation becomes

$$u(x,t) = (lnf)_{xx}. \tag{18}$$

We can write it in another form

$$u = w_{xx} \quad where \quad w = (lnf)_{xx}, \tag{19}$$

now from (19),we have

$$u_t = w_{xxt}, \quad u_{5x} = w_{7x}, \quad uu_{3x} = w_{2x}w_{5x}, \quad u_x u_{2x} = w_{3x}w_{4x} \quad and \quad u^2 u_x = w_{xx}^2 w_{3x},$$

putting the above expression into eq.(16),we get

$$w_{xxt} + w_{7x} + 30w_{2x}w_{5x} + 30w_{3x}w_{4x} + 180w_{xx}^2 w_{3x},$$

on integrating w.r.t x

$$w_{xt} + w_{6x} + 30(w_{2x}w_{4x}) + 60(w_{xx})^3 \tag{20}$$

then we calculate

$$\begin{aligned}
w_x &= \frac{f_x}{f}, \quad w_t = \frac{f_t}{f}, \\
w_{xx} &= \frac{f_{xx}f - f_x^2}{f^2}, \\
w_{xxxx} &= \frac{f_{xxxx}f^3 - 4f_{xxx}f_x f^2 + 6f_{xx}^2 f^2 + 12f_{xx}f_x^2 f - 6f_x^4}{f^4}, \\
w_{xxxxxx} &= \frac{1}{f^6}\Big[f_{xxxxxx}f^5 - 6f_{xxxxx}f_x f^4 + 15f_{xxxx}f_{xx}f^4 - 20f_{xxxx}f_x^2 f^3 + 15f_{xxx}^2 f^4 + 90f_{xxx}f_{xx}f_x f^3 \\
&\quad - 120f_{xxx}f_x^3 f^2 + 15f_{xx}^3 f^3 + 180f_{xx}^2 f_x^2 f^2 - 270f_{xx}f_x^4 f + 120f_x^6\Big].
\end{aligned}$$

Putting the all above values in equation (16),we get a billinear equation in f as

$$-f_t f_x + ff_{xt} - 10f^2{}_{3x} + 15f_{2x}f_{4x} - 6f_x f_{5x} + ff_{6x} = 0. \tag{21}$$

3.1.4 Sawada-Kotera(SK) equation

Taking SK equation [29] as

$$u_t + 5(uu_{2x})_x + 5u^2 u_x + u_{5x} = 0. \tag{22}$$

We suppose the phase variable $\eta_i = p_i x - q_i t$. On putting $u = e^{\eta_i}$ in eq.(22)

$$u_t = -q_i e^{\eta i},$$

$$u_{5x} = p_i^5 e^{\eta i},$$

putting in linear term of eq.(22)$u_t + u_{5x} = 0$, we get

$$-q_i e^{\eta i} + p_i^5 e^{\eta i} = 0,$$

$$e^{\eta i}(p_i^5) = q_i e^{\eta i},$$

Then we get dispersion relation as

$$q_i = p_i^5.$$

Taking the Cole-Hopf transformation as

$$u(x,t) = R(lnf)_{xx}, \tag{23}$$

for the given nonlinear PDE.

$$R\left(\left(\frac{p^2e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{\left(1+e^{-p^5t+px}\right)^2}\right)\right)_t + 5\left(R\left(\frac{p^2e^{-p^5t+px}}{1+e^{-p^5t+px}}\right)\right)$$
$$-\frac{p^2\left(e^{-p^5t+px}\right)^2}{\left(1+e^{-p^5t+px}\right)^2}\left(R\left(\frac{p^2e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{\left(1+e^{-p^5t+px}\right)^2}\right)\right)_{x,x}$$
$$+5R\left(\frac{p^2e^{-p^5t+px}}{1+e^{-p^5t+px}}\right)_{x,x} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{\left(1+e^{-p^5t+px}\right)^2}\left(R\left(\frac{p^2e^{-p^5t+px}}{1+e^{-p^5t+px}}\right)\right)_{x,x,x}$$
$$+5R^2\left(\frac{p^2e^{-p^5t+px}}{1+e^{-p^5t+px}}\right) - \frac{p^2\left(e^{-p^5t+px}\right)^2}{\left(1+e^{-p^5t+px}\right)^2}\left(\left(\frac{p^2e^{-p^5t+px}}{1+e^{-p^5t+px}} - \frac{p^2\left(e^{-p^5t+px}\right)^2}{\left(1+e^{-p^5t+px}\right)^2}\right)^2\right)_x$$
$$+\left(R\left(\frac{p^2e^{-p^5t+px}}{1+e^{-p^5t+px}}\right)\right)_{x,x,x,x,x} = 0$$

$$\Rightarrow \frac{-5\left((R-10)\,e^{-2p(p^4t-x)} + (-R+10)e^{-p(p^4t-x)} + 2e^{-3p(p^4t-x)} - 2\right)Rp^7e^{-2p(p^4t-x)}(R-6)}{\left(1+e^{-p(p^4t-x)}\right)^7 = 0}$$

$$\Rightarrow R = 6.$$

We get the value of R is 6 by putting the function f is $f(x,t) = 1 + e^{\eta_1}$ in equation(22).
So, the logarithmic transformation becomes

$$u(x,t) = 6(lnf)_{xx}, \tag{24}$$

we can write it in another form

$$u = w_{xx} \quad where \quad w = 6(lnf)_{xx}, \tag{25}$$

now from (25),we have

$$u_t = w_{xxt}, \quad u_x u_{xx} = w_{xxx}w_{xxxx}, \quad uu_{xxx} = w_{xx}w_{xxxxx} \quad and \quad u_{5x} = w_{7x},$$

putting the above expression into eq.(22),we get

$$w_{xxt}+5w_{3x}w_{4x}+5w_{2x}w_{5x}+5(w_{xx})^2w_{3x}+w_{7x}=0$$

on integrating w.r.t x

$$w_{xt}+(5w_{xx}w_{4x})+\frac{5}{3}(w_{xx})^3+w_{6x}=0, \tag{26}$$

then we calculate

$$\begin{aligned}
w_x &= \frac{6f_x}{f}, \quad w_t=\frac{6f_t}{f},\\
w_{xt} &= \frac{\partial}{\partial t}\left(\frac{6f_x}{f}\right)=6\left(\frac{f_{xt}f-f_xf_t}{f^2}\right)=\frac{6(f_{xt}f-f_xf_t)}{f^2},\\
w_{xx} &= \frac{\partial}{\partial x}\left(\frac{6f_x}{f}\right)=6\left(\frac{f_{xx}f-f_x^2}{f^2}\right)=\frac{6(f_{xx}f-f_x^2)}{f^2},\\
w_{xxxx} &= \frac{6(f_{xxxx}f^3-6f_{xxx}f_xf^2+6f_{xx}^2f^2+18f_{xx}f_x^2f-24f_x^4)}{f^4},\\
w_{xxxxxx} &= \frac{6}{f^6}\Big[f_{xxxxxx}f^5-15f_{xxxxx}f_xf^4+30f_{xxxx}f_{xx}f^4-60f_{xxxx}f_x^2f^3+60f_{xxx}^2f^4+180f_{xxx}f_{xx}f_xf^3\\
&\quad -360f_{xxx}f_x^3f^2+90f_{xx}^3f^3+540f_{xx}^2f_x^2f^2-1080f_{xx}f_x^4f+720f_x^6\Big].
\end{aligned}$$

Putting the all above values in equation(22),we get a billinear equation in f as

$$-f_tf_x+ff_{xt}-10{f^2}_{3x}+15f_{2x}f_{4x}-6f_xf_{5x}+ff_{6x}=0. \tag{27}$$

3.2 (2+1)-dimensional Equation

3.2.1 Kadomstsev Petviashvili (KP) Equation

We have the integrable KP equation [25] as

$$(u_t+6uu_x+u_{xxx})_x+u_{yy}=0. \tag{28}$$

We suppose the phase variable $\eta_i=p_ix+q_iy-d_it$. With

$$u=e^{\eta_i},$$

we have

$$\begin{aligned}
u_{tx} &= -d_ipe^{\eta i},\\
u_{xxxx} &= p_i^4e^{\eta i},\\
u_{yy} &= q_i^2e^{\eta i},
\end{aligned}$$

putting in linear term of eq.(28)$u_{tx}+u_{xxxx}+u_{yy}=0$,

$$-d_ipe^{\eta i}+p_i^4e^{\eta i}+q_i^2e^{\eta i}=0,$$

$$e^{\eta i}(p_i^4+q_i^2)=d_ip_ie^{\eta i},$$

we get dispersion relation

$$d_i=\frac{p_i^4-q_i^2}{p_i}.$$

Finding the Cole-Hopf transformation

$$u(x,y,t) = R(lnf)_{xx}, \tag{29}$$

for the given nonlinear PDE.

$$\left(R\left(\frac{p^2e^{px+qy-\frac{(p^4-q^2)t}{p}}}{1+e^{px+qy-\frac{(p^4-q^2)t}{p}}}-\frac{p^2\left(e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}{\left(1+e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}\right)\right)_{t,x}$$

$$+6R\left(\frac{p^2e^{px+qy-\frac{(p^4-q^2)t}{p}}}{1+e^{px+qy-\frac{(p^4-q^2)t}{p}}}-(\frac{p^2\left(e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}{\left(1+e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}\right)\left(R\left(\frac{p^2e^{px+qy-\frac{(p^4-q^2)t}{p}}}{1+e^{px+qy-\frac{(p^4-q^2)t}{p}}}-\frac{p^2\left(e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}{\left(1+e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}\right)\right)_{x,x}$$

$$+6\left(R\left(\frac{p^2e^{px+qy-\frac{(p^4-q^2)t}{p}}}{1+e^{px+qy-\frac{(p^4-q^2)t}{p}}}-\frac{p^2\left(e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}{\left(1+e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}\right)\right)_{x}$$

$$\left(+R\left(\frac{p^2e^{px+qy-\frac{(p^4-q^2)t}{p}}}{1+e^{px+qy-\frac{(p^4-q^2)t}{p}}}-\frac{p^2\left(e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}{\left(1+e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}\right)\right)_{x,x,x,x}$$

$$-\left(R\left(\frac{p^2e^{px+qy-\frac{(p^4-q^2)t}{p}}}{1+e^{px+qy-\frac{(p^4-q^2)t}{p}}}-\frac{p^2\left(e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}{\left(1+e^{px+qy-\frac{(p^4-q^2)t}{p}}\right)^2}\right)\right)_{y,y}=0$$

$$\frac{12Rp^6e^{\frac{-2p^4t+2p^2x+2qyp+2q^2t}{p}}\left(e^{\frac{-2p^4t+2p^2x+2qyp+2q^2t}{p}}-3e^{\frac{-p^4t+p^2x+qyp+q^2t}{p}}+1\right)(R-2)}{\left(1+e^{\frac{-p^4t+p^2x+qyp+q^2t}{p}}\right)^6}=0$$

$$\Rightarrow R = 2$$

We get the value of R is 2 by putting the function f is $f(x,y,t) = 1 + e^{\eta_1}$ in equation(28). So,

$$u(x,t) = 2(lnf)_{xx}. \tag{30}$$

We can write it in another form

$$u = w_{xx} \quad where \quad w = 2(lnf)_{xx}, \tag{31}$$

now from (31),we have

$$u_t = w_{xxt}, \quad u_x = w_{xxx} \quad and \quad u_{xxx} = w_{xxxxx}.$$

Putting the above expression into eq.(28),we get

$$(w_{xxt} + 6w_{xx}w_{xxx} + w_{xxxxx})_x - w_{xxyy} = 0$$

on integrating w.r.t x

$$w_{xxt} + 6w_{xx}w_{xxx} + w_{xxxxx} - w_{xyy} = 0, \tag{32}$$

on again integrating w.r.t x

$$w_{xt} + 6\int w_{xx}w_{xxx}\partial x + w_{xxxx} - w_{yy} = 0, \tag{33}$$

here taking

$$I = 6\int w_{xx}w_{xxx}\partial x = \frac{1}{2}\int 2w_{xx}w_{xxx}\partial x = \frac{1}{2}w_{xx}^2,$$

substituting the value of I in eq.(33), we get

$$w_{xt} + 3w_{xx}^2 + w_{xxxx} - w_{yy} = 0, \tag{34}$$

then we calculate

$$\begin{aligned}
w_x &= \frac{2f_x}{f},\\
w_{xt} &= \frac{2(f_{xt}f - f_xf_t)}{f^2},\\
w_{xx} &= \frac{2(f_{xx}f - f_x^2)}{f^2},\\
w_{xxx} &= \frac{2(f_{xxx}f^2 - 6f_{xx}f_xf + 4f_x^3)}{f^3},\\
w_{xxxx} &= \frac{2(f_{xxxx}f^3 - 8f_{xxx}f_xf^2 + 12f_{xx}^2f^2 + 24f_{xx}f_x^2f - 12f_x^4)}{f^4},\\
w_{yy} &= \frac{2(f_{yy}f - f_y^2)}{f^2}.
\end{aligned}$$

Putting the all above values in equation(28),we get a billinear equation in f as

$$ff_{xt} - f_xf_t + 3{f^2}_{xx} - 4f_xf_{xxxx} + ff_{xxxx} - ff_{yy} + f_y^2 = 0. \tag{35}$$

3.3 (3+1)-dimensional Equation

3.3.1 Generalised BKP Equation

We consider BKP equation [26] as

$$u_{yt} + 3u_{xz} - 3u_xu_{xy} - 3u_{2x}u_y - u(_{3x})_y = 0. \tag{36}$$

We suppose the phase variable $\eta_i = p_ix + q_iy + r_iz - d_it$. With

$$u = e^{\eta_i}$$

, we have

$$u_{yt} = -d_iq_ie^{\eta i},$$

$$u_{xz} = p_i r_i e^{\eta i},$$
$$u(_{3x})_y = p_i^3 q_i e^{\eta i},$$

putting in linear term of eq.(36) $u_{yt} + u_{xz} + u(_{3x})_y = 0$, then

$$-d_i q_i e^{\eta i} + 3p_i r_i e^{\eta i} - p_i^3 q_i e^{\eta i} = 0,$$

$$e^{\eta i}(-3p_i r_i + p_i^3 q_i) = d_i q_i e^{\eta i},$$

we get dispersion relation as

$$d_i = \frac{-3p_i r_i + p_i^3 q_i}{q_i}.$$

Taking the Cole-Hopf transformation

$$u(x,y,t) = R(lnf)_{xx}, \tag{37}$$

for the given nonlinear PDE.

$$\left(\frac{Rpe^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}{1+e^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}\right)_{t,y} + 3\left(\frac{Rpe^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}{1+e^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}\right)_{x,z}$$
$$-3\left(\frac{Rpe^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}{1+e^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}\right)_{x}\left(\frac{Rpe^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}{1+e^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}\right)_{x,y}$$
$$-3\left(\frac{Rpe^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}{1+e^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}\right)_{x,x}\left(\frac{Rpe^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}{1+e^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}\right)_{y}$$
$$-\left(\frac{Rpe^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}{1+e^{px+qy+rz-\frac{(p^3q+3pr)t}{q}}}\right)_{x,x,x,y} = 0,$$

$$\Rightarrow \frac{6Rp^4 e^{\frac{2q^2y+\left(2p^3t+2px+2rz\right)q-6prt}{q}}}{q\left(e^{\frac{q^2y+(p^3t+px+rz)q-3prt}{q}}-1\right)\left(1+e^{\frac{q^2y+(p^3t+px+rz)q-3prt}{q}}\right)^5}(R-2) = 0$$

$$R = 2$$

We get the value of R is 2 by putting the function $f(x,y,t) = 1 + e^{\eta_1}$ in equation(36) .
So, the logarithmic transformation becomes

$$u(x,t) = 2(lnf)_{xx}, \tag{38}$$

we can write it in another form

$$u = w_{xx} \quad where \quad w = 2(lnf)_{xx}, \tag{39}$$

So,

$$w_x = \frac{2\partial_x f}{f} = \frac{2f_x}{f},$$

$$w_y = \frac{2\partial_y f}{f} = \frac{2f_y}{f},$$

$$w_z = \frac{2\partial_z f}{f} = \frac{2f_z}{f},$$

$$w_t = \frac{2\partial_t f}{f} = \frac{2f_t}{f},$$

$$w_{yt} = \frac{2(f_{yt}f - f_y f_t)}{f^2},$$

$$w_{xz} = \frac{2(f_{xz}f - f_x f_z)}{f^2},$$

$$w_{xx} = \frac{2(f_{xx}f - f_x^2)}{f^2},$$

$$w_{xy} = \frac{2(f_{xy}f - f_x f_y)}{f^2},$$

$$w_{xxxy} = \frac{2}{f^4}\begin{pmatrix} f_{xxxy}f^3 - 3f_{xxx}f_y f^2 \\ -3f_{xxy}f_x f^2 + 6f_{xx}f_x f_y f \\ -2f_x^3 f_y \end{pmatrix}.$$

Putting the all above values in equation(36),we get a billinear equation in f as

$$-f_t f_y + f f_{yt} - 3f_x f_z + 3f f_{xz} - 3f_{xy}f_{2x} + f_x f({}_{2x})_y - f_y f_{3x} - f f({}_{3x})_y = 0. \tag{40}$$

4 Application of bilinear equations

Nonlinear partial differential equations (PDEs) play a crucial role in describing various physical phenomena in fluid dynamics, plasma physics, nonlinear optics, and soliton theory. One powerful method for solving and analyzing nonlinear PDEs is the bilinear method, especially in the form introduced by Hirota. Bilinear equations play an significant role in different thechniques such as Hirota method, simplified Hirota technique, Bäcklund transformation, direct symbolic approach, bilinear neural network method, and symbolic bilinear technique in solving nonlinear PDEs. The original nonlinear PDE is converted into a bilinear equation, which acts as a framework to search for solutions such as breathers, umps, kinks and rogue waves.
The Hirota approach, which includes the development of multi-soliton solutions, is a straightforward technique for determining the precise solutions of nonlinear PDEs. The dependent variable transformation is used to translate the provided PDE into bilinear equation. Bilinearization plays an essential role in many direct solution methods because it makes it possible to apply strong mathematical techniques to obtain exact solutions.
Tools for symbolic computation are frequently employed to carry out the bilinearization procedure and to handle the resulting analysis. Recent research focuses on algorithmic and symbolic computation approaches to automatically derive bilinear forms. These methods use computer algebra systems such as Mathematica, Maple, or Matlab to assist in the bilinearization of complex nonlinear systems.
In order to make the bilinear technique for (n+1)-dimensional PDEs more accessible and systematic, symbolic methods have been suggested to generalize it. In simple terms, bilinear equations serve as an intermediate

step between the original, usually complicated nonlinear equations, and more manageable forms that permit the determination of exact solutions. They are a key element in the study of integrable systems and soliton theory, as they allow researchers to employ advanced techniques for analyzing and solving these equations.

5 Conclusion

In this work, we constructed the bilinear equation for a class of (n + 1)-dimensional nonlinear partial differential equations (PDEs). We studied several well-known equations such as KdV, KP, CDG, SK, gKP equations and others, and formulated their bilinear equations. An important and developing field in the study of nonlinear PDEs is the bilinear approach. Its scope is expanding quickly into increasingly complicated systems with the integration of algorithmic tools and symbolic computation. Bilinearization is therefore essential to nonlinear science research, both theoretically and practically. The KdV equation, Boussinesq equaton, KP equation, SK equation, shallow water wave, modified BKP equation, and other well-known nonlinear equations were used to test this approach. The program Mathematica was used to solve these using our novel approach.
Our findings demonstrate the method's effectiveness and dependability in locating bilinear versions of nonlinear PDEs. Numerous disciplines, including nonlinear dynamics, oceanography, mathematical physics, fluid dynamics, and soliton theory, use these kinds of equations. As a result, our simplification is highly beneficial and highly suggested for further cutting-edge study and development.

Declarations

Ethics approval and consent to participate

Not applicable.

Conflict of interest

The author claims that there are no conflicts of interest.

Data availability statement

No datasets have been generated or analyzed during the current investigation.

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