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A Study for Analytical Solutions to Nonlinear Evolution Equations with Analytical Approach

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Abstract: This research presents an analytical study of the (3+1)-dimensional KdV-type nonlinear evolution equation using the Generalized Exponential Rational Function Method (GERFM). This approach is an effective analytical technique for obtaining analytical solutions of nonlinear partial differential equations arising in plasma physics, fluid dynamics, nonlinear optics, and wave propagation phenomena. The governing nonlinear equation is transformed into an ordinary differential equation through a travelling wave transformation. And, a homogeneous balance principle is applied to construct suitable exponential rational trial functions. By employing the GERFM approach, several exact analytical solutions are obtained and classified into different families representing soliton waves, kink-type waves, periodic structures, and singular wave solutions. It further analyzes the solutions graphically with two- and three-dimensional plots. This graphical analysis is carried out with *Mathematica* to study the wave propagation behavior, stability, localization, and interaction of nonlinear waves under appropriate parameters. The study confirms the efficiency and applicability of GERFM in investigating higher-dimensional nonlinear evolution equations and provides useful insights into multidimensional wave dynamics.

1. Introduction

With our current knowledge in general, we understand many processes occurring in the nature in our actual world, ranging from linear science to nonlinear science [2]. Within a linear system, input is proportional to the output. In most systems found in natural world yet these systems are nonlinear and they do not have such a straightforward connection between the input and the output. They are shaped using nonlinear partial differential equations (NPDEs) [11-12].

The NPDEs are essential to describing the nature of wave phenomena in systems as varied as plasmas and optics [7,13]. In many of these systems one finds the balance of nonlinearity and dispersion to give lead to phenomena such as stable solitary waves known as solitons [2,10]. Nonlinearity sharpens the wave while dispersion spreads the wave energy in space. Whenever these two phenomena reach a balance, stable structures form [2]. Aside from these there are also breather waves where amplitude changes frequently in time, periodic waves which are recurring in space and time [3,6] and the more destructive phenomenon of rogue waves which are the incredibly large waves known to arise in the ocean [4,6].

Many previous research is focused on one- or two-dimensional models as these are mathematically easy but in actuality the systems engaged are multi-dimensional. So, in the last few years multi-dimensional models are gone popular. Especially (3+1)-dimensions; 3 dimensions for space and 1 dimension for time. The multi-dimensional systems represent waves moving in the space regarding each other, not as the waves in only one or two dimensions of the space [2,6].

1.1 Classification of Nonlinear Wave Structures

One important work when examining NPDE is the classification of different wave structures produced. It is to note that even though the wave pattern forms are only mathematically determined, they do indeed represent a physical reality and can be classified according to their stability and the speed at which they propagate [3,11].

- *Solitons:* Solitons are steady waves which keep their shape and amplitude while traveling and do not disperse. These solitons. just keep on going and they do not change they stay the size and shape the whole time. They can interact with each other without losing their identity. Many examples are known in different areas of study such as surface and inner water waves and in optics and photonics [2,10].

- *Breathers:* The breathers are the structures, which is periodic in time. Breather has an oscillating structure in which the magnitude is going in and out. This causes the reason to have "breathing"-like properties [3,6].

- *Periodic Waves*: The same infinite waves which are reproductive of its shape in space and time will be obtained in an enclosed system, and a stationary propagation wave will always be present [3].

- *Rogue Waves:* Rogue .waves are highly unstable waves which occur suddenly and dissipate rapidly. Due to their large amplitudes, they are very dangerous in oceanography [4,6].

1.2 Analytical Frameworks and Symbolic Computation

In the past research, most of the works deals with seeking for approximated numerical solutions for nonlinear equations. The numerical methods do provide a useful information but do not always give us an adequate knowledge of the systems. However, the exact analytical solution can perfectly reveal the dynamics of wave phenomena [11].

The present research mainly concerns developing analytical methods systematically that can find various types of wave solutions. The methods using symbolic computation and direct analytical approaches are significant in studying nonlinear wave problems [3,11]. The Generalized Exponential Rational Function Method (GERFM) is one such effective way to convert nonlinear equation to the simplest one, seeking the exact solutions of the original nonlinear equations. With an appropriate transformation, nonlinear partial differential equation is converted to ordinary differential equation whose exact solution can be found [3,11] easily.

1.3 Visualizing Waves: Graphs and Figures

Visualisation is essential in the study of nonlinear wave behaviour. Mathematical solutions of the nonlinear wave equation are generally more comprehensible as graphs. Graphical representation aids in a clear depiction of the physical characteristics of wave structures [3].

3D Surface Plots: These plots show wave amplitude across the space it occurs over. They illustrate wave shapes and structures and in turn indicate where crests and troughs occur. These plots are valuable visualisations of changes in wave amplitude.

2D Contour Plots: They represent the phenomenon under the study of two dimensions, by mapping with colour or contour lines. The plots aid in visualizing wave propagation and interaction phenomena.

Physical interpretation: It becomes apparent what the results mean for actual physical phenomena as opposed to an abstract mathematical solution through the use of graphing techniques, which helps to explain the features that we observe in phenomena like localised solitons (humps of waves) and periodic waves (sinusoidal waves) respectively. It gives the researcher some intuition as to how stable and dynamic are these types of waves.

Overall objective: The principal aim of this study is to analyse nonlinear wave equations. We utilize analytical method and investigate the characteristics of a number of wave forms in higher dimensional spaces to form realistic mathematical models for a variety of physical phenomena [2,5].

2. The General Exponential Rational Function Method (GERFM)

The Generalized Exponential Rational Function Method (GERFM) for finding solutions to complex equations is utilized to study the nonlinear PDE. For solving the problems, the solution is considered as a rational function expressed in terms of exponentials in both numerator and denominator. This is highly effective for analyzing models of (3+1)-dimensions due to its complex algebra.

2.1 Steps in GERFM

***Step 1**: The Traveling Wave Transformation*
In this starting step, we transform the given Partial Differential Equation (PDE) to a simple ordinary differential equation (ODE). Transformation of the equation changes the original equation with many independent variables to a single independent variable and thus becomes very easier to solve.

***Step 2**: Choosing a Trial Function*
The real strength of GERFM comes from its initial guess. We suggest the answer is a fraction, both numerator and denominator of which consist of exponential functions. They are given coefficients which allow us to vary them so the math can form many different shapes, capable of identifying peaks, through to a repeating wave form.

***Step 3**: The Homogeneous Balancing Principle*
Before we can find the final answer, we have to find the balancing number as a theoretical test to make sure the forces that aim to sharpen the wave perfectly matched by the (non-linearity) forces trying to spread it out (dispersion). By comparing the nonlinear and the highest derivative terms of the equation, we find the balancing number.

***Step 4**: Solving the Algebraic Map*
Now we substitute our assumed solution back into the equation, which will generate a huge equation. For the solution to be actually a solution we need every term within this equation, no matter how huge, to be zero. This provides a set of equations in the form of algebra that are very tedious to solve by hand, it is here where symbolic software such as *Mathematica* comes in, to help us solve the equations to provide our values for the constants.

***Step 5**: Sorting the Physical Results*
- The final step is to take those numbers and create actual structures. Depending on what the constants are doing, one of these fields is likely applicable.
- Soliton solutions are solutions describing "bounded", "travelling" waves, which do not disperse.
- Periodic solutions are solutions that repeat endlessly.
- Rational solutions sometimes describe "rogue" waves, or sudden large events where a height difference can occur.

2.2 Mathematical Description of GERFM

To find the analytic solutions of the (3+1)-dimensional KdV type equation, we utilize a well known generalized exponential rational function method (GERFM). This method can be explored generally into steps as

- Let us consider a (3+1)-dimensional nonlinear partial differential equations (PDE)

$$P\left(u, u_x, u_y, u_z, u_t, u_{xx}, \dots\right) = 0 \tag{1}$$

and apply the traveling wave transformation

$$u(x, y, z, t) = G(\rho), \qquad \rho = sx + ky + rz + jt + \lambda\,,$$

then the studied nonlinear PDE (1) converts into an ordinary differential equation (ODE)

$$Q(G, G', G'', G''', \dots) = 0 \tag{2}$$

- We suppose the solution of the equation (2) as

$$G(\rho) = H_0 + \sum_{i=1}^{n} \; H_i N(\rho)^i + \sum_{i=1}^{n} \; L_i N(\rho)^{-i} \tag{3}$$

where n is the balancing constant obtained by using homogeneous balance principle, and $N(\rho)$ is a rational function

$$N(\rho) = \frac{\omega_1 e^{n_1 \rho} + \omega_2 e^{n_2 \rho}}{\omega_3 e^{n_3 \rho} + \omega_4 e^{n_4 \rho}}\,, \tag{4}$$

with arbitrary constants ω_i, η_i and constant coefficients $\mathrm{H}_0, \mathrm{H}_\mathrm{i}$ and K_i.

- On substituting the equation (3) with (4) into the equation (2), collecting all the possible powers of { e^{ρ} }, and equating their coefficients C_h for the integer h to zero, forms an algebraic system $C_h = 0$.

- Finally, once we have solved the system of equations, we take the values we discovered and plug them back into our formulas (3) and (4) to lock in the solutions for our simplified ODE. Then, we perform a back substitution that is where we move those results back into the original KdV-type equation. This last step allows us to create the final, exact mathematical answer that describes how waves move and behave in a (3+1)-dimensional space.

Our particular deal of a KdV-type equation for this chapter, we are looking at this because it deals with 3 spatial directions x y, z and also 1 dimension in time t. The value of this model comes from describing real phenomena. The real-world processes described by the KdV-type equation model such as internal gravity waves, which are waves found deep within oceans where the ocean is of different densities vertically and ion-acoustic waves, which exist in certain plasmas used to study energy and space.

3. Investigation of the (3+1)-dimensional KdV-type Equation

The investigation of nonlinear evolution equations falls within mathematical physics. This allows us to learn how waves move in various settings. The Korteweg-de Vries (KdV) equation is a very famous example of a nonlinear evolution equation. The original equation of this form was published in 1895 to model the motion of long shallow waves on water. Initially the KdV equation was a (1+1)-dimensional model. This implies that the waves we are looking at only move in one direction.

In reality waves are generally not traveling in one single direction. Instead, they are traveling in many different directions. So, when they collide their effects may be observed as much greater or lesser than those seen in (1+1) dimensional space. For this reason, we need to get a more realistic study of wave propagation. The KdV equation has been extended into greater dimensions as to KP equation. The KdV equation and its many variations have been put to use in studying the motion of many waves in the literature. The KdV equation is a subject that is widely used by many scientists today to determine various things about wave motion.

With the KdV-type equation model we are able to see how these waves remain so stable when travelling over. distances and through their respective environment with the third spatial dimension included. The KdV-type equation model helps us to see nature rather than being surrounded by nothing but math. The fact that the KdV-type equation model deals with 3 directions of space and 1 direction of time is how we have progressed in understanding the complex wave behaviors.

3.1 Mathematical Structure and Terminology

The KdV-type equation that investigated is a complex equation that models the progression and amplification of waves over time. The KdV-type equation is used by professionals to examine the behaviour of waves over a four-dimensional space (width, length, height and time).

The KdV-type equation investigates three spatial directions (width, length, height) and time. By investigating these dimensions at a particular time, it enables us to understand how the chaotic movement of complex waves can become stable.

The main equation is defined as:

$$U_{xxxy} + \alpha_1 U_{yt} + \alpha_2 \left(U_x U_y\right)_x + \alpha_3 U_{xx} + \alpha_4 U_{zz} = 0 \tag{5}$$

To investigate the wave, we have to break down each variable to gain an understanding of how it acts both as a rule and a force upon the wave.

- **The High-Order Dispersion Term U_{xxxy}**
  Term U_{xxxy} is one part of the math describing how the wave is expanding while moving. With this it's means the wave starts spreading while moving and this is called dispersion, where U_{xxxy} can be said to represent the dispersion of the wave. This term is what makes KdV-type equations special because this is where the x-direction is connected to the side-to- side direction. U_{xxxy} makes sure the KdV-type equations can form waves because it

acts as a counter balancing term making sure the wave is not forming too high of a peak and then crashing. If it weren't for the counter balancing term the KdV-type equation would not be able to maintain its shape. The term U_{xxxy} is essential for the KdV-type equation to be able to produce waves that maintain their stability.

- **The Temporal Evolution Term $\boldsymbol{\alpha_1 U_{yt}}$**

The term $\alpha_1 U_{yt}$ describes how the waves evolve as time continues to pass by. There is one term which serves as balancing coefficient between the sideways stretching of the wave and time it is experiencing. This makes this model unique by giving it ability to tell us what the wave is predicted to do in the future by examining the current wave state.
Here, U_{yt} describes the future development of the wave with time. This coefficient balances how much the wave spreads to the sides and how fast the time proceeds. In other words, it suggests a predictable nature of the wave over time.

- **The Nonlinear Interaction Term $\boldsymbol{\alpha_2 (U_x U_y)_x}$**

The term $\alpha_2 (U_x U_y)_x$ plays. a vital role in ensuring the wave maintains its shape. When split up, this term expresses the interaction between different parts of the wave. This term, when expressed, causes the wave to steepen up as the high parts of the wave travel at different speed compared to the shallow parts. This phenomenon would normally cause the wave to break. In this case there is a perfectly balanced pair of effects, allowing the wave to propagate great distances. The term $(U_x U_y)_x$ and a second term offset each other. The $(U_x U_y)_x$ term is crucial in the formation of solitons or steady waves.

- **The Transverse Stability Terms ($\boldsymbol{\alpha_3 U_{xx}}$ *and* $\boldsymbol{\alpha_4 U_{zz}}$)**

These order spatial derivatives are like the forces that keep the wave stable and stop it from falling apart. The wave is traveling through a medium. These forces are what pull the wave back into its original shape.
There are two parts to this:

- This $\alpha_3 U_{xx}$ part is what handles the stability of the wave as it moves in its main direction.
- This $\alpha_4 U_{zz}$ part is what makes this model special because it accounts for changes in the height or depth of the wave. This means the equation can describe waves moving through a three-dimensional space, like the deep ocean or a huge field of plasma not just a flat two-dimensional surface.

KdV-type equation is unique because two forces are in balance with each other. This wave is trying to make itself steeper. The wave is also trying to spread out. In these three models the balancing between the forces .is more complex as it occurs in all three dimensions simultaneously. This balance allows the wave to retain its shape while it travels through space. The objective is to use the Generalized Exponential Rational Function Method to solve for α_1 ,$\alpha_2, \alpha_3, \alpha_4$ values that will create a solitary wave. A solitary wave is a wave packet of energy which can transmit over a long distance without changing its shape. The Generalized Exponential Rational Function Method can also find the exact parameters values.

The KdV-type equation is special as it can be used to show the Soliton Equilibrium. Soliton Equilibrium occurs when the wave has balanced itself out perfectly, such that it retains its shape as it travels. The $\alpha_{1\,,}, \alpha_2, \alpha_3, \alpha_4$ are key in that they have balanced the wave. As the wave propagates .through the medium the $\alpha_{1\,,}, \alpha_2, \alpha_3, \alpha_4$ parameters are responsible in maintaining its shape. The Generalized Exponential Rational Function Method provides the ability to discover these values.

3.2 Solutions of (3+1)-dimensional KdV-type equation

This study aims to derive various exact analytical solutions for the (3+1)-dimensional KdV-type equation such as solitons, kink-type waves, breathers, and periodic structures. By applying the GERFM protocol described above to the governing equation, we initiate the following analytical process:

Considering the wave transformation

$$u(x, y, z, t) = G(\rho), \qquad \rho = sx + ky + rz + jt + \lambda\,, \tag{6}$$

where s, k, r and λ are arbitrary constants. On substituting the equation (6) into (5), we get a transformed equation in the form of an ordinary differential equation (ODE) as

$$s^3 k G^{(4)} + (\alpha_1 kj + \alpha_3 s^2 + \alpha_4 r^2) G'' + 2\alpha_2 s^2 k G' G'' = 0. \tag{7}$$

Balancing the highest-order derivative term ($G^{(4)}$) with the highest-order nonlinear term ($G' G''$) gives $n + 4 = (n + 1) + (n + 2)$, hence the balancing constant $n = 1$. Hence, from equation (3), we get the trial solution as

$$G(\rho) = H_0 + H_1 N(\rho) + \frac{L_1}{N(\rho)}, \tag{8}$$

where $N(\rho)$ is rational function as in equation (4). Next, we substitute the equation (8) into (7) and follow the steps of the GERFM. To obtain the different solutions, we consider different families for different values of the constants in the rational function (4).

Family 1: For $[\omega_1, \omega_2, \omega_3, \omega_4] = [5,1,1,1]$ and$[\eta_1, \eta_2, \eta_3, \eta_4] = [1,0,1,0]$, then the equation (4) becomes

$$N(\rho) = \frac{1+5e^{\rho}}{1+e^{\rho}}. \tag{9}$$

On substituting equation (9) into (8), we get

$$G(\rho) = H_0 + \frac{(1+5e^{\rho})H_1}{1+e^{\rho}} + \frac{(1+e^{\rho})K_1}{1+5e^{\rho}}\,. \tag{10}$$

On putting the equation (10) with (9) into the equation (7). On solving the obtained system, we get values as

Case 1.1 $H_1 \to \frac{3s}{2\alpha_2},\ L_1 \to 0,\ k \to \frac{-s^2\alpha_3 r^2\alpha_4}{s^3 j\alpha_1}$

Substituting the values of the above constants into equation (10), we get a solution for (7) as

$$G[\rho] \to \frac{3(1+5e^{\rho})s}{2(1+e^{\rho})\alpha_2} + H_0.$$

Consequently, an analytic solution of (5) is obtained as

$$G[j\,t + sx + ky + rz + \lambda] \to \frac{3(1+5e^{jt+sx+ky+rz+\lambda})s}{2(1+e^{jt+sx+ky+rz+\lambda})\alpha_2} + H_0\,. \tag{11}$$

Case 1.2 $L_1 \to -\frac{15s}{2\alpha_2},\ H_1 \to 0,\ k \to \frac{-s^2\alpha_3 r^2\alpha_4}{s^3 j\alpha_1}$

Substituting the values of the above constants into equation (10), we get a solution for (7) as

$$G[\rho] \to -\frac{15(1+e^{\rho})s}{2(1+5e^{\rho})\alpha_2} + H_0.$$

Consequently, an analytic solution of (5) is obtained as:

$$G[j\,t + sx + ky + rz + \lambda] \to -\frac{15(1+e^{jt+sx+ky+rz+\lambda})s}{2(1+5e^{jt+sx+ky+rz+\lambda})\alpha_2} + H_0. \tag{12}$$

Case 2. $H_1 \to \frac{3s}{2\alpha_2},\ L_1 \to 0,\ k \to \frac{-ks^3 - s^2\alpha_3 - r^2\alpha_4}{k\alpha_1}$

Substituting the values of the above constants into equation (10), we get a solution for (7) as

$$G[\rho] \to \frac{3(1+5e^{\rho})s}{2(1+e^{\rho})\alpha_2} + H_0.$$

Consequently, an analytic solution of (5) is obtained as

$$G[j\,t + sx + ky + rz + \lambda] \to \frac{3(1+5e^{jt+sx+ky+rz+\lambda})s}{2(1+e^{jt+sx+ky+rz+\lambda})\alpha_2} + H_0. \tag{13}$$

Family 2: For $[\omega_1, \omega_2, \omega_3, \omega_4] = [1,1,-1,1]$ and$[\eta_1, \eta_2, \eta_3, \eta_4] =$ [-1,-1,-1,0], then the equation becomes

$$N(\rho) = \frac{2e^{-\rho}}{1-e^{-\rho}}. \tag{14}$$

On substituting equation (14) into (8), we get

$$G(\rho) = H_0 + \frac{2e^{-\rho} H_1}{1-e^{-\rho}} + \frac{1}{2} e^{\rho}(1-e^{-\rho})L_1. \tag{15}$$

On putting the equation (15) with (14) into the equation (7). On solving the obtained system, we get values as

Case 1. $H_1 \to \frac{3s}{\alpha_2},\ L_1 \to 0,\ j \to \frac{-ks^3-\alpha_3 s^2-r^2\alpha_4}{k\alpha_1}$

Substituting the values of the above constants into equation (15), we get a solution for (7) as

$$G[\rho] \to \frac{6e^{-\rho}s}{(1-e^{-\rho})\alpha_2} + H_0.$$

Consequently, an analytic solution of (5) is obtained as

$$G[j\,t + sx + ky + rz + \lambda] \to \frac{6e^{-(jt+sx+ky+rz)}s}{(1-e^{-(jt+sx+ky+rz)})\alpha_2} + H_0. \tag{16}$$

Case2. . $H_1 \to \frac{3s}{\alpha_2},\ L_1 \to 0,\ j \to \frac{-s^2\alpha_3-r^2\alpha_4}{s^3+j\alpha_1}$

Substituting the values of the above constants into equation (15), we get a solution for (7) as

$$G[\rho] \to \frac{6e^{-\rho}s}{(1-e^{-\rho})\alpha_2} + H_0.$$

Consequently, an analytic solution of (5) is obtained as

$$G[j\,t + sx + ky + rz + \lambda] \to \frac{6e^{-(jt+sx+ky+rz)}s}{(1-e^{-(jt+sx+ky+rz)})\alpha_2} + H_0. \tag{17}$$

Family 3: For $[\omega_1, \omega_2, \omega_3, \omega_4] = [3,2,1,1]$ and $[\eta_1, \eta_2, \eta_3, \eta_4] = [1,1,1,0]$, then the equation (4) becomes

$$N(\rho) = \frac{5e^{\rho}}{1+e^{\rho}} \tag{18}$$

On substituting equation (18) into (8), we get

$$G(\rho) = H_0 + \frac{5e^{\rho}H_1}{1+e^{\rho}} + \frac{1}{5}e^{-\rho}(1+e^{\rho})L_1. \tag{19}$$

On putting the equation (19) with (18) into the equation (7). On solving the obtained system, we get values as

Case 1. . $H_1 \to \frac{6s}{5\alpha_2},\ L_1 \to 0,\ j \to \frac{-ks^3-\alpha_3 s^2-r^2\alpha_4}{k\alpha_1}$

Substituting the values of the above constants into equation (19), we get a solution for (7) as

$$G[\rho] \to \frac{6e^{\rho}s}{(1+e^{\rho})\alpha_2} + H_0.$$

Consequently, an analytic solution of (5) is obtained as

$$G[j\,t\ + sx + ky + rz + \lambda] \to \frac{6e^{sx+ky+rz+\frac{t(-ks^3-\alpha_3 s^2-r^2\alpha_4\,)}{k\alpha_1}+\lambda}s}{\left(1-e^{sx+ky+rz+\frac{t(-ks^3-\alpha_3 s^2-r^2\alpha_4\,)}{k\alpha_1}+\lambda}\right)\alpha_2} + H_0\ . \tag{20}$$

Case 2. . $H_1 \to \frac{6s}{5\alpha_2},\ L_1 \to 0,\ j \to \frac{-ks^3-\alpha_3 s^2-r^2\alpha_4}{k\alpha_1}$

Substituting the values of the above constants into equation (19), we get a solution for (7) as

$$G[\rho] \to \frac{6e^{\rho}s}{(1+e^{\rho})\alpha_2} + H_0.$$

Consequently, an analytic solution of (5) is obtained as

$$G[j\,t\ + sx + ky + rz + \lambda] \to \frac{6e^{(jt+sx+ky+rz)}s}{\left(1-e^{(jt+sx+ky+rz)}\right)\alpha_2} + H_0. \tag{21}$$

Family 4: For $[\omega_1, \omega_2, \omega_3, \omega_4] = [2,2,2,-3]$ and$[\eta_1, \eta_2, \eta_3, \eta_4] = [2,\text{-}2,2,\text{-}2]$, then the equation (4) becomes

$$N(\rho) = \frac{2e^{-2\rho}+2e^{2\rho}}{-3e^{-2\rho}+2e^{2\rho}}. \tag{22}$$

On substituting equation (18) into (8), we get

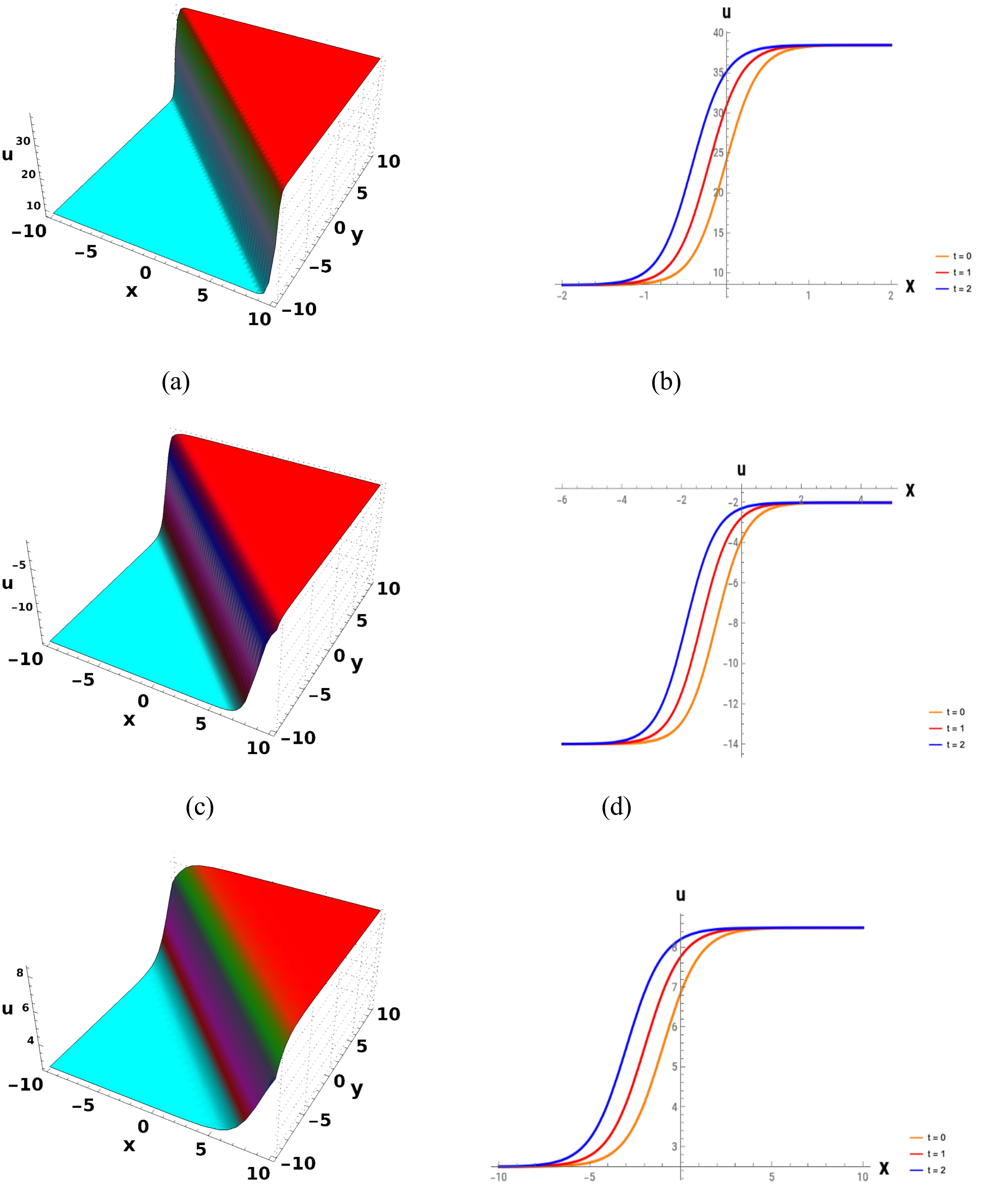
$$G(\rho) = H_0 + \frac{5e^{\rho}H_1}{1+e^{\rho}} + \frac{1}{5}e^{-\rho}(1+e^{\rho})L_1. \tag{23}$$

On putting the equation (23) with (22) into the equation (7). On solving the obtained system, we get values as

Case 1.1 $H_1 \to 0,\ L_1 \to \frac{48s}{5\alpha_2},\ j \to \frac{-16ks^3-\alpha_3 s^2-r^2\alpha_4}{k\alpha_1}$

Substituting the values of the above constants into equation (23), we get a solution for (7) as

$$G[\rho] \to \frac{48\left(-3e^{-2\rho}+2e^{2\rho}\right)s}{5\left(2e^{-2\rho}+2e^{2\rho}\right)\alpha_2} + H_0.$$

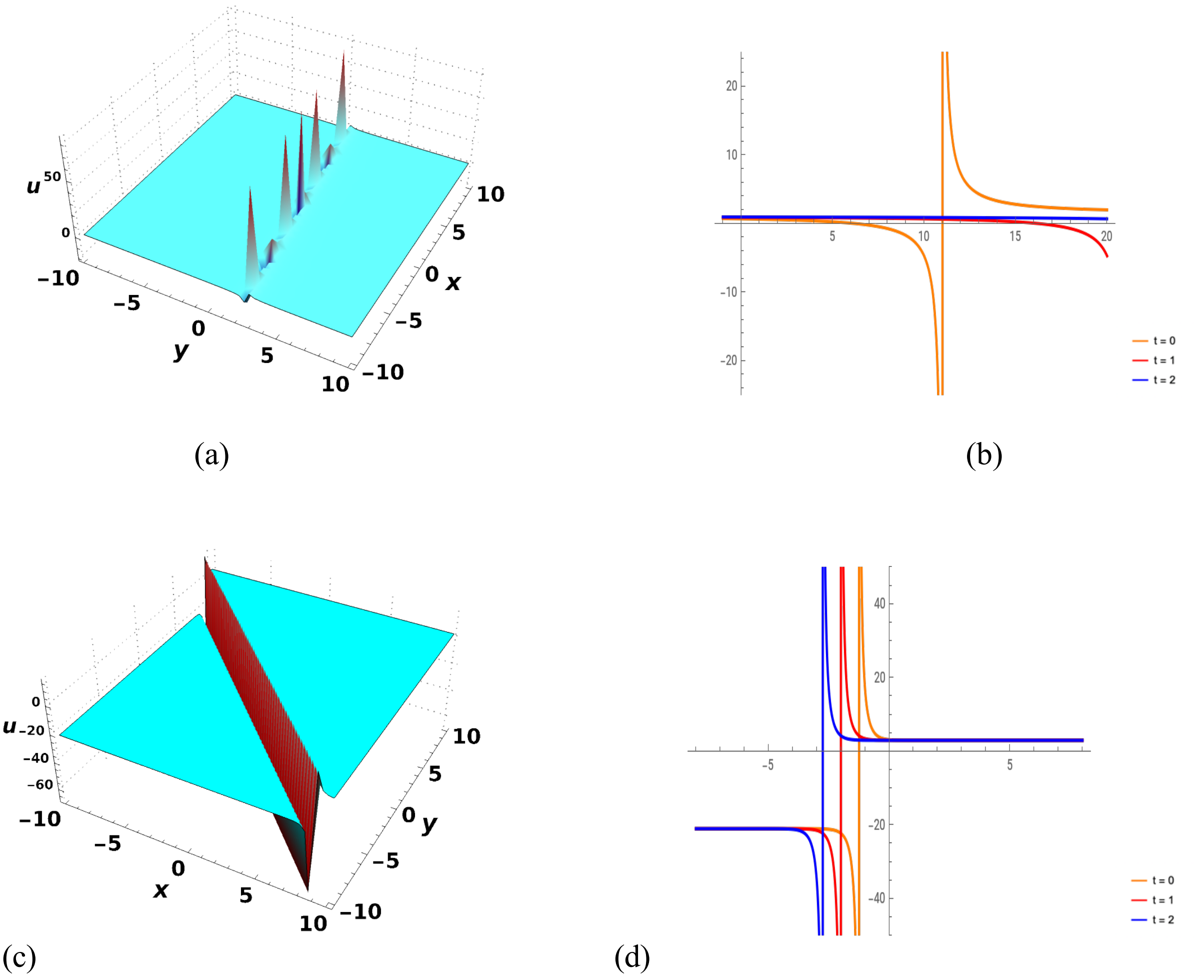


(a) (b)

(c) (d)

(e) (f)

Figure 1: Graphics for the solutions (11), (12), and(13) as in (a), (c) and (e) respectively, with their 2D plots in (b), (d) and (f), having parameters: $(a)\ t = 0, z = 0, j = 1\ , H_0 = 1, s = 5,$ $k = 5, r = 1,\ \alpha_2 = 1, \lambda = 0.1\ ; (b)\ t = 0,\ z = 0,\ j = 1\ , H_0 = 1, s = 2,\ k = 2, r = 1,\ \alpha_2 = 1,\ \lambda = 1.1; (c)\ t = 0, z = 0, j = 1\ , H_0 = 1, s = 1,\ k = 1, r = 1,\ \alpha_2 = 1, \lambda = 1$



(a) (b)

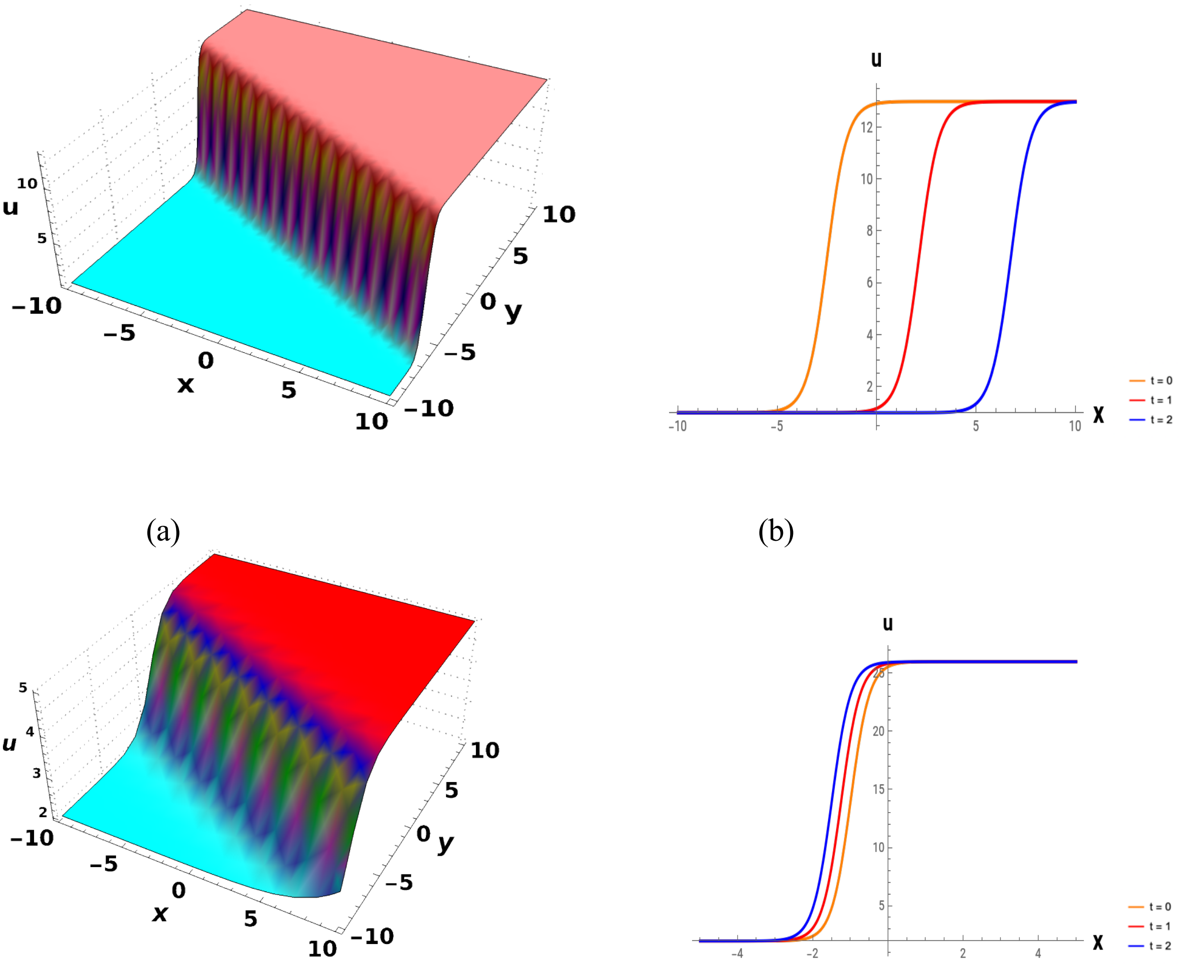
(c) (d)

Figure 2: Graphics for the solutions (16) and (17) as in (a) and (c) respectively, with their 2D plots in (b) and (d), having parameters: $(a) t = 1, z = 0, j = 1 , H_0 = 1, s = -.1,\ k = -.8, r = 1,\ \alpha_2 = 1, \lambda = 0.1$; $(b)\ t = 1, z = 0,\ j = 3 , H_0 = 3, s = 4, k = 4, r = 1,\ \alpha_2 = 1, \lambda = 2.3$

Consequently, an analytic solution of (5) is obtained

$$G[\rho] \to \frac{48\left(-3e^{-2(jt+sx+ky+rz)}+2e^{2(jt+sx+ky+rz)}\right)s}{5\left(2e^{-2(jt+sx+ky+rz)}+2e^{2(jt+sx+ky+rz)}\right)\alpha_2} + H_0\ . \tag{24}$$

Case 1.2 $H_1 \to \frac{72s}{5\alpha_2},\ L_1 \to 0,\ j \to \frac{-16ks^3-\alpha_3 s^2-r^2\alpha_4}{k\alpha_1}$



(a) (b)

(c) (d)

Figure 3: Graphics for the solutions (20) and (21) as in (a) and (c) respectively, with their 2D plots in (b) and (d),having parameters ; $(a) t = 0, z = 0, j = 1\,, H_0 = 1, s = 2,\ k = 4, r = 1,\ \alpha_1 = 1,\ \alpha_2 = 1\,,\ \alpha_3 = 1,\ \alpha_4 = 1\,, \lambda = 5\,; (b)\ t = 0,\ z = 0,\ j = 1\,, H_0 = 2, s = .5,\ k = 1, r = 1,\ \alpha_2 = 1,\ \lambda = 4$

Substituting the values of the above constants into equation (23), we get a solution for (7) as

$$G[\rho] \rightarrow \frac{72\left(2e^{-2\rho}+2e^{2\rho}\right)s}{5\left(-3e^{-2\rho}+2e^{2\rho}\right)\alpha_2} + H_0\,.$$

Consequently, an analytic solution of (5) is obtained

$$G[\rho] \rightarrow \frac{72\left(2e^{-2(jt+sx+ky+rz)}+2e^{2(jt+sx+ky+rz)}\right)s}{5\left(-3e^{-2(jt+sx+ky+rz)}+2e^{2(jt+sx+ky+rz)}\right)\alpha_2} + H_0\,. \tag{25}$$

Case 2. $$H_1 \to \frac{72s}{5\alpha_2},\ L_1 \to 0,\ j \to \frac{-s^2\alpha_3 - r^2\alpha_4}{16s^3 + j\alpha_1}$$

Substituting the values of the above constants into equation (23), we get a solution for (7) as

$$G[\rho] \to \frac{72\left(2e^{-2\rho} + 2e^{2\rho}\right)s}{5\left(-3e^{-2\rho} + 2e^{2\rho}\right)\alpha_2} + H_0\ .$$

Consequently, an analytic solution of (5) is obtained

$$G[\rho] \to \frac{72\left(2e^{-2(jt+sx+ky+rz)} + 2e^{2(jt+sx+ky+rz)}\right)s}{5\left(-3e^{-2(jt+sx+ky+rz)} + 2e^{2(jt+sx+ky+rz)}\right)\alpha_2} + H_0\ . \tag{26}$$

Family 5: For $[\omega_1, \omega_2, \omega_3, \omega_4] = [1,2,3,-1]$ and$[\eta_1, \eta_2, \eta_3, \eta_4] = [2,-2,2,-2]$, then the equation (4) becomes

$$N(\rho) = \frac{2e^{-2\rho} + e^{2\rho}}{-e^{-2\rho} + 3e^{2\rho}}. \tag{27}$$

On substituting equation (18) into (8), we get

$$G(\rho) = H_0 + \frac{2e^{-2\rho} + e^{2\rho}}{-e^{-2\rho} + 3e^{2\rho}} H_1 + \frac{\left(-e^{-2\rho} + 3e^{2\rho}\right)}{2e^{-2\rho} + e^{2\rho}} L_1. \tag{28}$$

On putting the equation (28) with (27) into the equation (7). On solving the obtained system, we get the values as

Case 1. $$H_1 \to 0,\ L_1 \to \frac{48s}{7\alpha_2},\ j \to \frac{-16ks^3 - \alpha_3 s^2 - r^2\alpha_4}{k\alpha_1}$$

Substituting the values of the above constants into equation (28), we get a solution for (7) as

$$G[\rho] \to \frac{48\left(-e^{-2\rho} + 3e^{2\rho}\right)s}{7\left(2e^{-2\rho} + e^{2\rho}\right)\alpha_2} + H_0.$$

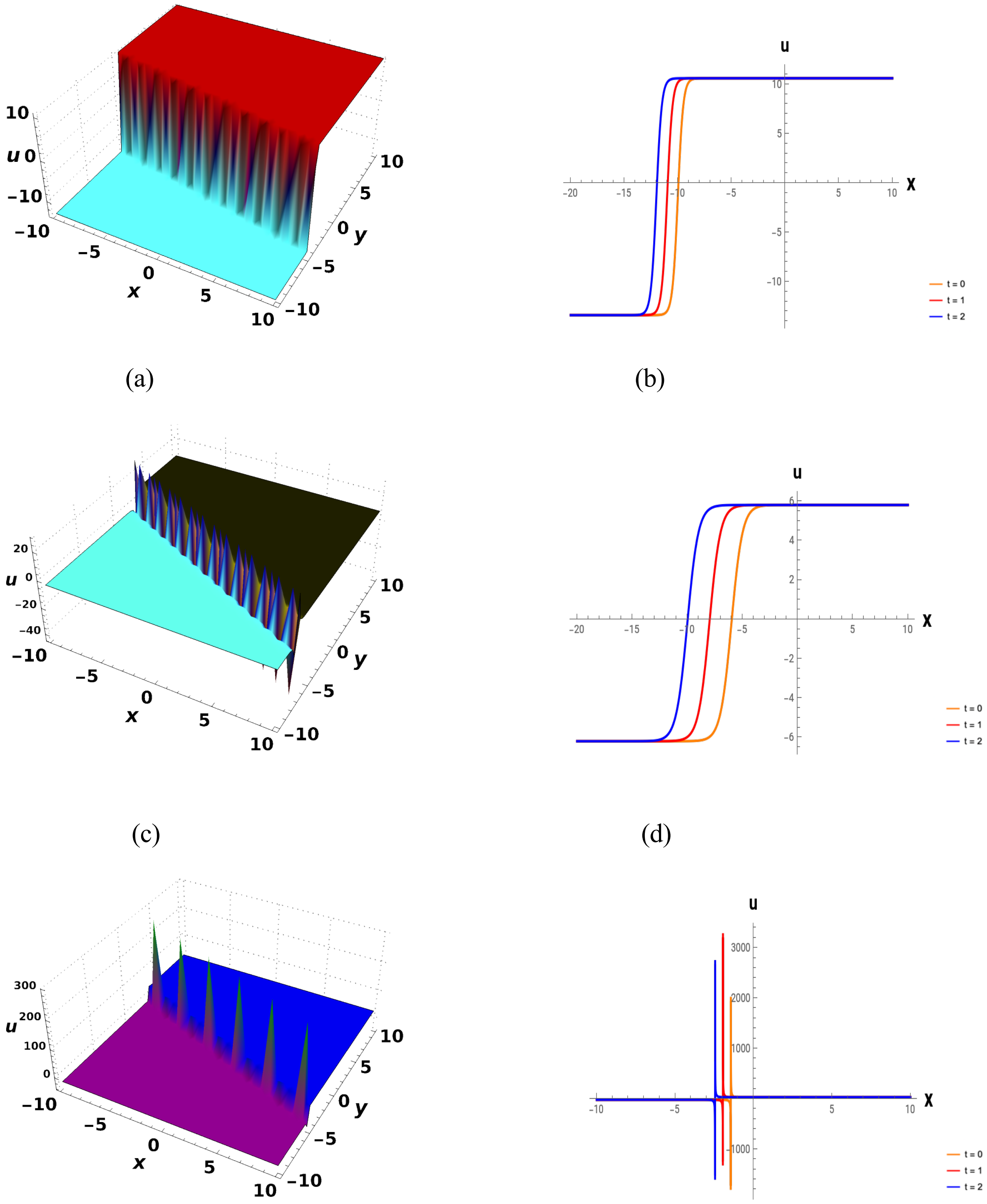
Consequently, an analytic solution of (5) is obtained

$$G[\rho] \to \frac{48\left(-e^{-2(jt+sx+ky+rz)} + 3e^{2(jt+sx+ky+rz)}\right)s}{7\left(2e^{-2(jt+sx+ky+rz)} + e^{2(jt+sx+ky+rz)}\right)\alpha_2} + H_0\ . \tag{29}$$

Case 2. $$H_1 \to \frac{72s}{7\alpha_2},\ L_1 \to 0,\ k \to \frac{-s^2\alpha_3 - r^2\alpha_4}{16s^3 + j\alpha_1}$$

Substituting the values of the above constants into equation (28), we get a solution for (7) as

$$G[\rho] \to \frac{72\left(2e^{-2\rho} + e^{2\rho}\right)s}{7\left(-e^{-2\rho} + 3e^{2\rho}\right)\alpha_2} + H_0.$$



(a) (b)

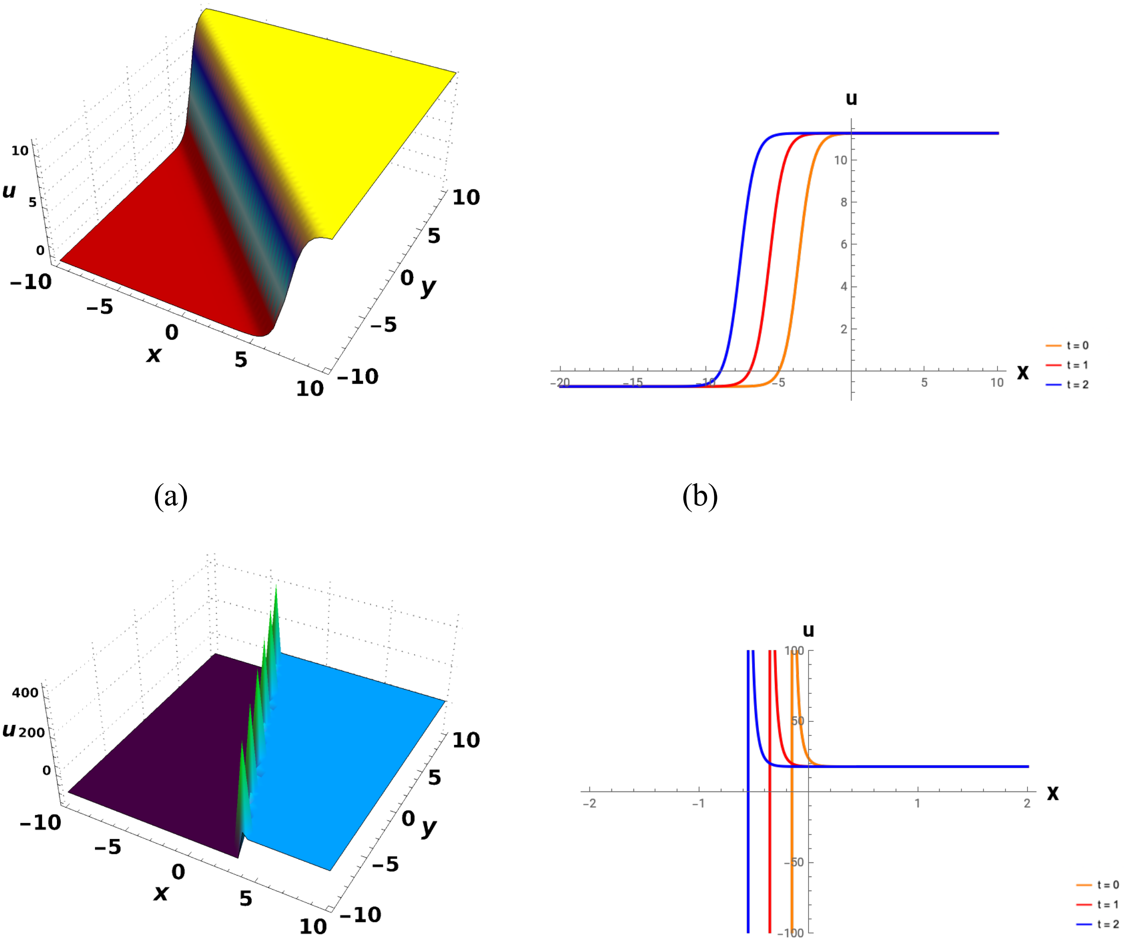
(c) (d)

(e) (f)

Figure 4: Graphics for the solutions (24),(25) and (26) as in (a) ,(c) and (e) respectively, with their 2D plots in (b),(d) and (f),having parameters; $(a)\ t = 1, z = 0, j = 1\ , H_0 = 1, s = 1,$ $k = 5, r = 1,\ \alpha_2 = 1, \lambda = 9\ ; (b)\ t = 0,\ \ z = 0,\ \ j = 1\ , H_0 = 1, s = .5,\ \ k = 1, r = 1,\ \alpha_2 =$ $1,\ \lambda = 2; (c)\ t = 0, z = 0, j = 1\ , H_0 = 1, s = 2,\ \ k = 5, r = 1,\ \alpha_2 = 1, \lambda = 3$

Consequently, an analytic solution of (5) is obtained

$$G[\rho] \rightarrow \frac{72\left(2e^{-2(jt+sx+ky+rz)}+e^{2(jt+sx+ky+rz)}\right)s}{7\left(-e^{-2(jt+sx+ky+rz)}+3e^{2(jt+sx+ky+rz)}\right)\alpha_2} + H_0 \,. \tag{30}$$



(a) (b)

(c) (d)

Figure 5: Graphics for the solutions (29) and (30) as in (a) and (c) respectively, with their 2D plots in (b) and (d),having parameters; $(a)\ t = 1, z = 0, j = 1\,, H_0 = 1, s = .5,\ k = .5, r = 1,\ \alpha_2 = 1, \lambda = 0.5\,; (b)\ t = 0,\ z = 0,\ j = 1\,, H_0 = 1, s = 5,\ k = 2, r = 1,\ \alpha_2 = 1,\ \lambda = 0.5$

4. Result Analysis and Discussions

In this work, different analytical solutions of the (3+1)-dimensional KdV-type equation are obtained using the GERFM method. This method uses a trial function with some adjustable parameters. By changing these parameters, different types of solutions are found. These solutions show different wave forms such as solitons, kink waves, periodic waves, and some sharp wave structures. It shows how nonlinearity and dispersion affect the behavior of waves.

Nonlinearity and dispersion have an impact on waves. To understand these solutions clearly, we need to look at some graphs that were made using *Mathematica*. The 3D graphs of waves show that the waves move in space and the 2D graphs of waves show how the wave changes over time. We can see how nonlinearity and dispersion affect the behaviour of waves by looking at these graphs of waves. The 2D plots are taken at different time values to study the behaviour of the waves.

From the graphs, it is seen that most waves keep their shape while moving, which means they are stable. Also, small changes in parameters can change the type of wave formed. The analysis of figures are given as:

Figure 1 shows the dynamical analysis for the different solution cases of Family 1. Plots (a) and (c) depict the single solitons moving. in the positive direction of the x-axis, while plot (e) represents the kink-type soliton propagating in the negative direction of the x-axis. The 3D plots are presented in the (x, y)-coordinates, and 2D plots are shown with respect to the different time values as t = 0, 1, and 2.

In **Figure** 2, we depict the dynamical analysis for the different solution cases of Family 2. Plots (a) and (c) depict the single solitons moving in the negative direction of the x-axis. The 3D plots are presented in the (x, y)-coordinates, and the corresponding 2D plots (b) and (d) are illustrated for different time values t=1, 2, and 3.

Figure 3 shows the dynamical analysis for the different solution cases of Family 3. Plots (a) and (c) depict periodic-type soliton structures propagating along the x-axis, where oscillatory wave patterns can be observed. The 3D plots are presented in the (x, y)-coordinates, and the corresponding 2D plots (b) and (d) are illustrated for different time values t=0, 1, and 2.

In **Figure** 4, we represent the dynamical analysis for the different solution cases of Family 4. Plots (a), (c), and (e) depict periodic and singular-type soliton structures propagating along the x-axis, where sharp oscillations and steep wave fronts are observed. The 3D plots are presented in the (x, y)-coordinates, and 2D plots are shown with respect to the different time values as (b) t = 1, 2, 3, (d) t = 1, 2, 3 and (f) t = 0, 1, 2.

Figure 5 depicts the dynamical analysis for the different solution cases of Family 5. Plots (a) and (c) depict kink-type and singular soliton structures propagating along the x-axis, where smooth transitions as well as sharp variations in amplitude are observed. The 3D plots are presented in the (x, y)-coordinates, and 2D plots are shown with respect to the different time values as (b) t = 1, 2, 3, and (d) t = 0, 1, 2.

5. Conclusions

In this study, the (3+1)-dimensional KdV-type equation has been investigated using the

Generalized Exponential Rational Function Method (GERFM). By applying a wave transformation, the given partial differential equation is reduced to an ordinary differential equation, which makes it easier to find analytical solutions. Different exact solutions are obtained by choosing different values of the parameters in the trial function. The obtained solutions are expressed in the form of exponential and rational functions and are grouped into different families. These solutions show various types of wave structures such as solitons, kink waves, periodic waves, and some sharp or singular wave forms. To understand how these solutions work, we utilized *Mathematica* to create the graphics for the obtained solutions. The 3D plots show how the waves move around in space and the 2D plots on the other hand show how the wave changes over time. The waves are clearly seen by looking at the shapes and motion from the graphs. It can be seen from what we have worked out that dispersion and nonlinearity are components in how these waves are formed and that the solitons will be stable and do not alter in shape as they propagate. The equation is useful to study waves when dealing with a nonlinear system and can be applied to a number of branches of physics and mathematics.

In the future, the GERFM method can be extended to analyze other nonlinear equations such as the KdV-type equations, the modified KdV-type equations, the nonlinear Schrodinger equation, the Burgers equation and the Boussinesq equation. The GERFM method can also be used to solve high-dimensional linear partial differential equations. Further work can investigate more of the role of parameters and new exact solutions can be sought for the GERFM method. Comparisons of GERFM method solutions and numerical methods may also be of use to understand more about the GERFM method. The results of this work using the GERFM method can be have application in many fields such as fluid dynamics, plasma physics, nonlinear optics and wave propagation in complex media.

Declarations

Ethics approval and consent to participate

Not applicable.

Competing interests

There is no conflict of interest, according to the authors.

Authors' contributions

Each author made an equal contribution to the final draft of the work. The authors would have consented and approved the final work.

Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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