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Nonlinear waves propagation for a generalized KdV-type evolution equation: solitons, breathers and their interactions

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Abstract: In this research, we analytically investigate the solutions of a generalized nonlinear higher-order Korteweg–de Vries (KdV)-type partial differential equation, which contains the Sawada-Kotera equation as its particular form. In the process of finding the analytical solutions, generalized exponential rational function (GERF) method is employed as the solution technique, in which the investigated nonlinear PDE is transformed into a simplified ordinary differential equation (ODE) utilizing a traveling wave transformation. A rational trial function involving exponential function terms is then considered for the reduced simple equation to derive the analytical solutions. We consider several families of parameters to analyse distinct solutions for the families of the trial function. The obtained solutions suggest the presence of a rich variety of nonlinear wave structures, such as bright and dark solitons, breathers, and oscillatory periodic background waves. The dynamical behavior of the solutions is analysed graphically using the symbolic system *Mathematica,* with appropriate choices of parameters. The wave propagation and its physical behaviour are illustrated in 2D and 3D graphics. The investigated equation has physical applications in applied mathematics, plasma physics, fluid dynamics, soliton theory, and other nonlinear sciences.

1 Introduction

Nonlinear partial differential equations (NLPDEs) are the significant tools in applied mathematics. NLPDEs in theoretical and applied sciences describe the interactions between dispersion and nonlinearity. This interaction affects a wide spectrum of wave propagation in physical media. These NLPDEs allow the self-reinforcing effects, resonance occurrences, and the spontaneous creation of coherent structures. Nonlinear wave propagation [16] is described in several equations such as the Schrödinger equation [3,4],the Boussinesq equation [21], the Sawada-Kotera and Lax equations. Also, these equations play a pivotal role in plasma physics, mechanical vibrations, and soliton theory [16, 18]. Solitory waves preserve their shape, amplitude, and speed during the propagation [11, 13]. Their applications can be seen in tsunami dynamics, fluid mechanics, optical fiber, plasma physics [15, 17].

1.1 Literature Review

The analytical solution of NLPDEs shows the precise geometric and temporal structure of the nonlinear wave models. These solutions explain the relation between dispersion and nonlinearity [10, 14]. The mathematical difficulties of solving complex NLPDEs have developed several approaches such as the Hirota bilinear method [15–18], the Bäcklund transformation [5, 6], the Darboux transformation [3, 4], the simplified Hirota method [1] and symbolic bilinear technique [7], the bilinear neural network method (BNNM) [8, 9] and direct symbolic approach (DSA) [10–14]. All of these techniques have expanded the range of solution classes. Different unique trial functions have been taken for obtaining wave structures. Many of these approaches have several limitations such as the Hirota method is computationally progresive but requires Painlev'e integrability test. Darboux and Bäcklund transformations become unwieldy for generalized equations with variable or arbitrary coefficients. Hence, we need a versatile, transparent, and diverse approach that can produce a variety of exact solution families.

The Generalized Exponential Rational Function Method (GERFM), introduced and advanced by Ghanbari, Osman, and collaborators [22–24], addresses precisely this need. The GERFM constructs the trial solution of the reduced ordinary differential equation (ODE) as a rational combination of exponential functions with arbitrary real parameters. By systematically varying the parameter vectors it generates distinct families of trial functions, each yielding a variousclass of exact solutions. The method requires only a traveling wave transformation to reduce the governing PDE to an ODE, application of the homogeneous balance principle to fix the truncation order N, and algebraic solution of the resulting coefficient system steps that are entirely transparent and implementable in standard symbolic computation software such as *Mathematica* . Compared with the Hirota technique, the GERFM does not require integrability as a prerequisite, and compared with the BNNM, it operates without neural network architecture or training procedures. The method is relatively new and has been explored for only a limited number of equations to date indicating substantial scope for further application and novel discovery.

1.2 Aim of the study

The present research work aims to investigate analytical propagation wave solutions for a generalized nonlinear fifth-order KdV-type partial differential equation [16]

$$u_t + A\,u^2 u_x + B\,u_x u_{xx} + C\,u\,u_{xxx} + u_{xxxxx} = 0, \tag{1}$$

where $u = u(x, t)$ is the wave amplitude function depending on the spatial coordinate x and temporal coordinate t, and A, B, C are real constant coefficients related to the physical parameters of the medium. In equation (1), the term u_t represents the temporal evolution of the wave profile; the term Au^2u_x is a higher-order (cubic) nonlinear convection term that models the interaction of a wave with itself at elevated amplitudes; the term $Bu_x u_{xx}$ captures the interplay between first- and second-order spatial gradients, contributing to wave-steepening and nonlinear dispersion; the term Cuu_{xxx} couples the wave amplitude directly with third-order spatial dispersion, a hallmark of KdV-type dynamics; and the fifth-order dispersive term u_{xxxxx} introduces higher-order dispersive effects that balance the combined nonlinear contributions to sustain stable solitary wave structures."

Equation (1) was originally construct by Kumar S. and Mohan B. in [16] as generalized nonlinear fifth-order KdV-type equation. The equation was obtained by using the recursion operator that was employed to the standard KdV hierarchy. It encompasses, as particular cases, two of the most important integrable fifth-order KdV equations: the Sawada–Kotera (SK) equation [16], obtained for $A = 5$, $B = 5$, $C = 5$ (i.e. $\lambda = 1$), and the Lax equation [16], obtained for $A = 30$, $B = 20$, $C = 10$ (i.e. $\lambda = 6$). Both the SK and Lax equations are completely integrable systems that admit multi-soliton solutions and have established applications in ion-acoustic wave propagation in plasma physics, nonlinear vibrations in mechanical engineering, shallow-water wave dynamics, and soliton theory [16]. The Painlevé integrability of equation (1) was demonstrated by Kumar and Mohan [16] through a Laurent series analysis about a singular manifold, confirming the existence of a full set of resonance conditions and the compatibility of arbitrary functions at each resonance. The Hirota bilinear approach was subsequently applied in [16] to establish an one-to-four soliton solution. In this process a universal logarithmic transformation obtained by Lagrange interpolation was used and the rich dynamical features of these solutions were graphically shown.As a result, the equation(1)plays a crucial role in the hierarchy of KdV-type evolution equations. This provides a suitable, physically motivated target for the application of innovative solution techniques. Investigating this equation with the GERFM it is possible to find solution classes that are inaccessible through the multi-soliton formalism. Additionally outside the scope of the Hirota bilinear approach, such as breathers, oscillatory solitons, and periodic background waves. .

1.3 Objectives

Main objective of this research work is application of the Generalized Exponential Rational Function Method (GERFM) to examine the nonlinear dynamical behavior of a generalized fifth-order Korteweg–de Vries (KdV)-type problem. Nonlinear equations of this type play a pivotal role in modeling intricate physical phenomena that are arising in nonlinear optics, fluid dynamics, plasma physics, and other practical disciplines where significant nonlinearity and higher-order dispersion coexist. The main goal of this work is to use the flexibility of the GERFM to generate a wide range of exact analytical solutions in closed form. The work aims to develop multiple solution structures that correlate to different physical wave behaviors. By introducing a rational trial function with numerous arbitrary parameters and methodically taking into account various parametric families. Specifically, various solution situations are obtained by exploring six different families of parameter sets, which enhances the governing equation's solution space.
Another important goal of this work is analysing the qualitative and dynamical characteristics of the solutions produced by graphical plots. Two-dimensional time-evolution wave profiles and three-dimensional surface plots are plotted for different parameter values. The objective of graphical analysis is to recognize and categorize different nonlinear wave structures, including kink-type waves, periodic wave trains, breather-like oscillations, bright solitons, dark solitons, and singular wave profiles, as well as their stability and propagation properties.

Additionally, this work is intends how the GERFM is a strong analytical tool that can solve higher-order nonlinear partial differential equations without the need for stringent integrability constraints. By doing this, it advances exact solution methods and lays the groundwork for additional theoretical and applied research in nonlinear wave theory.

Overall this research work objectives is connecting the acquired mathematical solutions with actual wave phenomena seen in a variety of science and engineering disciplines, this research work seeks to close the gap between analytical solution development and physical interpretation. .

Manuscript structure is given as: Section 2 provides a self-contained summary of the GERF approach, which covers ever algorithmic step from the traveling wave transformation through to back-substitution. In section 3 GERFM is applied to equation (1). The homogeneous balancing is performed, the trial solution is achieved, and the analytical solutions derived from six distinct families of parameter are shown. The graphics of obtained solutions were plotted in 3D and 2D using *Mathematica* software. In section 4, we analyze the overall result of the current research project additionally have a discussion part about individual graphical structures analysis of each family. Finally, section 5 concludes the work with a summary of findings and remarks on the physical significance and scope of the obtained solutions. Having established the broader context, identified the governing equation, and outlined the manuscript structure, we now turn to a detailed description of the Generalized Exponential Rational Function Method (GERFM) that will be employed throughout this work.

2 Methodology : Generalized Exponential Rational Function Method

To find the analytical solutions of the (1+1)- Nonlinear partial differential equation, we utilize a well-known generalized exponential rational function method (GERFM) [22–24]. This method can be generally described in the following steps:

- Let us consider a (1+1)-dimensional nonlinear partial differential equations (PDEs)

$$D(u, u_x, u_t, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxx} \cdots) = 0, \tag{2}$$

  and apply the traveling wave transformation $u(x,t) = G(\xi)$ where $\xi = sx + jt + \lambda$, then the studied Nonlinear PDE (2) converts into an ordinary differential equation (ODE)

$$F(G, G^{'}, G^{"}, G^{'''} \ldots) = 0. \tag{3}$$

- We suppose the solution of the equation (3) as

$$G(\xi) = H_0 + \sum_{i=1}^{N} H_i M(\xi)^i + \sum_{i=1}^{N} L_i M(\xi)^{-i} \tag{4}$$

  where N is the balancing constant obtained by using homogeneous balance principle, and $M(\xi)$ is a rational function

$$M(\xi) = \frac{w_1 e^{\eta_1 \xi} + w_2 e^{\eta_2 \xi}}{w_3 e^{\eta_3 \xi} + w_4 e^{\eta_4 \xi}}, \tag{5}$$

  with w_i and η_i are used as arbitrary constants with the value of i in range of $(1 \leq i \leq 4)$ and constant coefficients H_0, L_i and H_i where value of i is lie in the range of $(1 \leq i \leq N)$.

- On substituting the equation (4) with (5) into the equation (3), collecting all the possible powers of $\{e^{\xi}\}$, and equating their coefficients C_j for the integer j to zero, forms an algebraic system $C_j = 0$.

- At the end, after solving the system of equations, we will substitute the obtained values into the equations (4) and (5) that establishes the analytical solutions of the ODE (3). Further doing back substitution, we create the analytical solution for the investigated KdV-type equation (1).

3 Investigation of analytic solutions for KdV-type equation

The study in the work aims to find different types of analytical solutions for the studied nonlinear KdV-type equation (1) through various forms as soliton, kink-type soliton, lump-chain, breather and periodic background waves. Now utilizing the GERFM to the studied equation as per discussed in above section, we have following process as

Considering the wave transformation

$$u(x,t) = G(\xi); \quad \xi = sx + jt + \lambda, \tag{6}$$

where s, k and λ are arbitrary constants. On substituting the equation (6) into (1), we get a transformed equation in the form of an ordinary differential equation (ODE) as

$$jG'(\xi) + As\, G^2(\xi)G'(\xi) + Bs^3\, G'(\xi)G''(\xi) + Cs^3\, G(\xi)G^{(3)}(\xi) + s^5\, G^{(5(\xi)} = 0. \tag{7}$$

With the help of homogeneous balancing principle, we balance the terms $G^{(5)}$ and G^2G'' of the equation (7), we deduce $N+5 = 3N+1$, this calculation led to value of $N = 2$. Hence, from the equation (4), trial solution can be obtained as

$$G(\xi) = H_0 + H_1 M(\xi) + H_2 M(\xi)^2 + \frac{L_1}{M(\xi)} + \frac{L_2}{M(\xi)^2}, \tag{8}$$

where $M(\xi)$ is rational function as in equation (5). Next, we substitute the equation (8) into (7) and follow the steps of the GERFM. All obtained solutions were verified by direct substitution into the original equation using symbolic computation.
To obtain the various solutions, we consider different families for different values of the constants in the rational function (5).

Family 1: For $[w_1, w_2, w_3, w_4] = [-3, 5, 6, -2]$ and $[\eta_1, \eta_2, \eta_3, \eta_4] = [1, 0, 1, 0]$, then the equation (5) becomes

$$M(\xi) = \frac{5 - 3e^{\xi}}{-2 + 6e^{\xi}}. \tag{9}$$

On substituting equation (9) into (8), we get

$$G(\xi) = H_0 + \frac{H_1(5 - 3e^{\xi})}{(-2 + 6e^{\xi})} + \frac{H_2(5 - 3e^{\xi})^2}{(-2 + 6e^{\xi})^2} + \frac{L_1(-2 + 6e^{\xi})}{(5 - 3e^{\xi})} + \frac{L_2(-2 + 6e^{\xi})^2}{(5 - 3e^{\xi})^2}. \tag{10}$$

On putting the equation (10) with (9) into the equation (7), and collecting all the possible powers of $T_i = (\mathrm{e}^{\xi})^{\mathrm{i}}$for some integer i, forms an algebraic system $T_i = 0$ for all i. On solving the obtained system we get values.

Case 1.1:

$$H_1 = 0,\ H_2 = 0,\ j = \frac{1}{8}\left(12s^5 - \frac{B^2s^5}{A} - \frac{2Bcs^5}{A} + \frac{Bs^3\sqrt{-\left(s^4\left(40A - B^2 - 4Bc - 4c^2\right)\right)}}{A}\right).$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi) = \frac{19}{16}(\phi) + \frac{45}{16}\frac{\left(-2 + 6e^{\xi}\right)}{\left(5 - 3e^{\xi}\right)}(\phi) + \frac{\left(-2 + 6e^{\xi}\right)^2}{1024A\left(5 - 3e^{\xi}\right)^2}(\alpha) \tag{11}$$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(sx+jt+\lambda) = \frac{19}{16}(\phi) + \frac{45}{16}\frac{\left(-2+6e^{sx+jt+\lambda}\right)}{\left(5-3e^{sx+jt+\lambda}\right)}(\phi) + \frac{\left(-2+6e^{sx+jt+\lambda}\right)^2}{1024A\left(5-3e^{sx+jt+\lambda}\right)^2}(\alpha) \tag{12}$$

where

$$\alpha = -1200Bs^2 - 2400cs^2 + \sqrt{(1200Bs^2+2400cs^2)^2 - 57600000As^4}$$

$$\phi = (\frac{\sqrt{-\left(s^4\left(40A-B^2-4Bc-4c^2\right)\right)}}{A} - \frac{Bs^2}{A} - \frac{2cs^2}{A}$$

Case 1.2:

$$L_1 = 0,\ L_2 = 0,\ j = \frac{1}{8}\left(12s^5 - \frac{B^2s^5}{A} - \frac{2Bcs^5}{A} - \frac{Bs^3\sqrt{-\left(s^4\left(40A-B^2-4Bc-4c^2\right)\right)}}{A}\right).$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi) = \tfrac{19}{16}(\phi) + \tfrac{9}{4}\tfrac{\left(5-3e^{\xi}\right)}{\left(-2+6e^{\xi}\right)}(\phi) + \tfrac{\left(5-3e^{\xi}\right)^2}{4A(-2+6e^{\xi})^2}(\alpha)$$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(sx+jt+\lambda) = \frac{19}{16}(\phi) + \frac{9}{4}\frac{\left(5-3e^{sx+jt+\lambda}\right)}{\left(-2+6e^{sx+jt+\lambda}\right)}(\phi) + \frac{\left(5-3e^{sx+jt+\lambda}\right)^2}{4A(-2+6e^{sx+jt+\lambda})^2}(\alpha) \tag{13}$$

where

$$\phi = -\frac{\sqrt{-\left(s^4\left(40A-B^2-4Bc-4c^2\right)\right)}}{A} - \frac{Bs^2}{A} - \frac{2cs^2}{A}$$

$$\alpha = (-3Bs^2 - 6cs^2 - \sqrt{(3Bs^2+6cs^2)^2 - 360As^4})$$

Family 2: For $[w_1, w_2, w_3, w_4] = [8, -3, 4, 5]$ and $[\eta_1, \eta_2, \eta_3, \eta_4] = [0, 1, 0, 1]$, then equation (5) becomes,

$$M(\xi) = \frac{8-3e^{\xi}}{4+5e^{\xi}} \tag{14}$$

Next, we substitute equation (14) into (8) and we get:

$$H_0 + \frac{H_1(8-3e^{\xi})}{(4+5e^{\xi})} + \frac{H_2(8-3e^{\xi})^2}{(4+5e^{\xi})^2} + \frac{L_1(4+5e^{\xi})}{(8-3e^{\xi})} + \frac{L_2(4+5^{\xi})^2}{(8-3e^{\xi})^2} \tag{15}$$

Case 2.1:

$$H_1 = 0,\ H_2 = 0,\ j = \frac{1}{8}\left(\frac{Bs^3\sqrt{-\left(s^4\left(40A-B^2-4Bc-4c^2\right)\right)}}{A} - \frac{B^2s^5}{A} - \frac{2Bcs^5}{A} + 12s^5\right)$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi) = \frac{191}{676}(\phi) + \frac{126\left(4+5e^{\xi}\right)}{169\left(8-3e^{\xi}\right)}(\pi) + \frac{\left(4+5e^{\xi}\right)^2}{57122A\left(8-3e^{\xi}\right)^2}(\alpha).$$

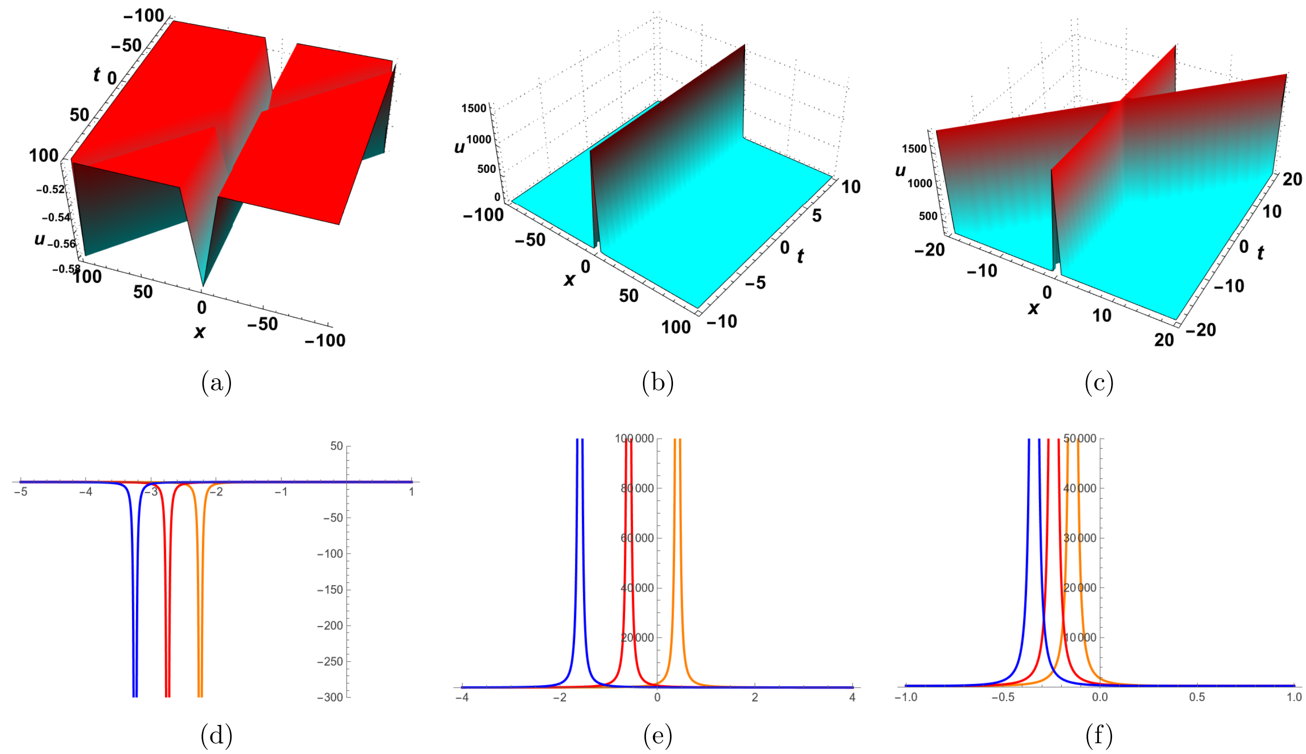

Figure 1: Graphics for the solutions (12) (a) and (b) for different parameter values and (13)(c) respectively, with their 2D plots in (d), (e), and (f), having parameters : (a) $t = 0$, $j = 1$, $s = 2$, $\lambda = 5$, $H_0 = 1$,$A = 3$,$B = 100$,$C = -30$; (b) $t = 0$, $j = 1$, $s = 1$, $\lambda = 0.1$, $H_0 = 1$,$A = 7$,$B = -400$,$C = 38$; (c) $t = 0$, $j = 1$, $s = 10$, $\lambda = 0.3$, $H_0 = 1$,$A = -1$,$B = 2$,$C = 1$;

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(sx + jt + \lambda) = \frac{191}{676}(\phi) + \frac{126\left(4 + 5e^{sx+jt+\lambda}\right)}{169\left(8 - 3e^{sx+jt+\lambda}\right)}(\pi) + \frac{\left(5e^{sx+jt+\lambda} + 4\right)^2}{57122A\left(8 - 3e^{sx+jt+\lambda}\right)^2}(\alpha) \tag{16}$$

where

$$\phi = \left(-\frac{\sqrt{\left(s^4\left(40A - B^2 - 4Bc - 4c^2\right)\right)}}{A} + \frac{Bs^2}{A} + \frac{2cs^2}{A}\right)$$

$$\pi = \left(\frac{\sqrt{-\left(s^4\left(40A - B^2 - 4Bc - 4c^2\right)\right)}}{A} - \frac{Bs^2}{A} - \frac{2cs^2}{A}\right)$$

$$\alpha = \left(-36504Bs^2 - 73008cs^2 + \sqrt{\left(36504Bs^2 + 73008cs^2\right)^2 - 53301680640As^4}\right)$$

Case 2.2:

$$L_1 = 0,\ L_2 = 0,\ j = \frac{1}{8}\left(-\frac{Bs^3\sqrt{-\left(s^4\left(40A - B^2 - 4Bc - 4c^2\right)\right)}}{A} - \frac{B^2s^5}{A} - \frac{2Bcs^5}{A} + 12s^5\right)$$

.

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi) = \frac{191}{676}(\phi) + \frac{105\left(8-3e^{\xi}\right)}{169\left(4+5e^{\xi}\right)}(\phi) + \frac{\left(8-3e^{\xi}\right)^2}{57122A\left(5e^{\xi}+4\right)^2}(\alpha)$$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(sx+jt+\lambda) = \frac{191}{676}(\phi) + \frac{105\left(8-3e^{sx+jt+\lambda}\right)}{169\left(5e^{sx+jt+\lambda}+4\right)}(\phi) + \frac{\left(8-3e^{sx+jt+\lambda}\right)^2}{57122A\left(5e^{sx+jt+\lambda}+4\right)^2}(\alpha) \tag{17}$$

where ,

$$\phi = \left(\frac{\sqrt{-\left(s^4\left(40A-B^2-4Bc-4c^2\right)\right)}}{A} + \frac{Bs^2}{A} + \frac{2cs^2}{A}\right)$$

$$\alpha = \left(-25350Bs^2 - 50700cs^2 - \sqrt{\left(-25350Bs^2+50700cs^2\right)^2 - 25704900000As^4}\right)$$

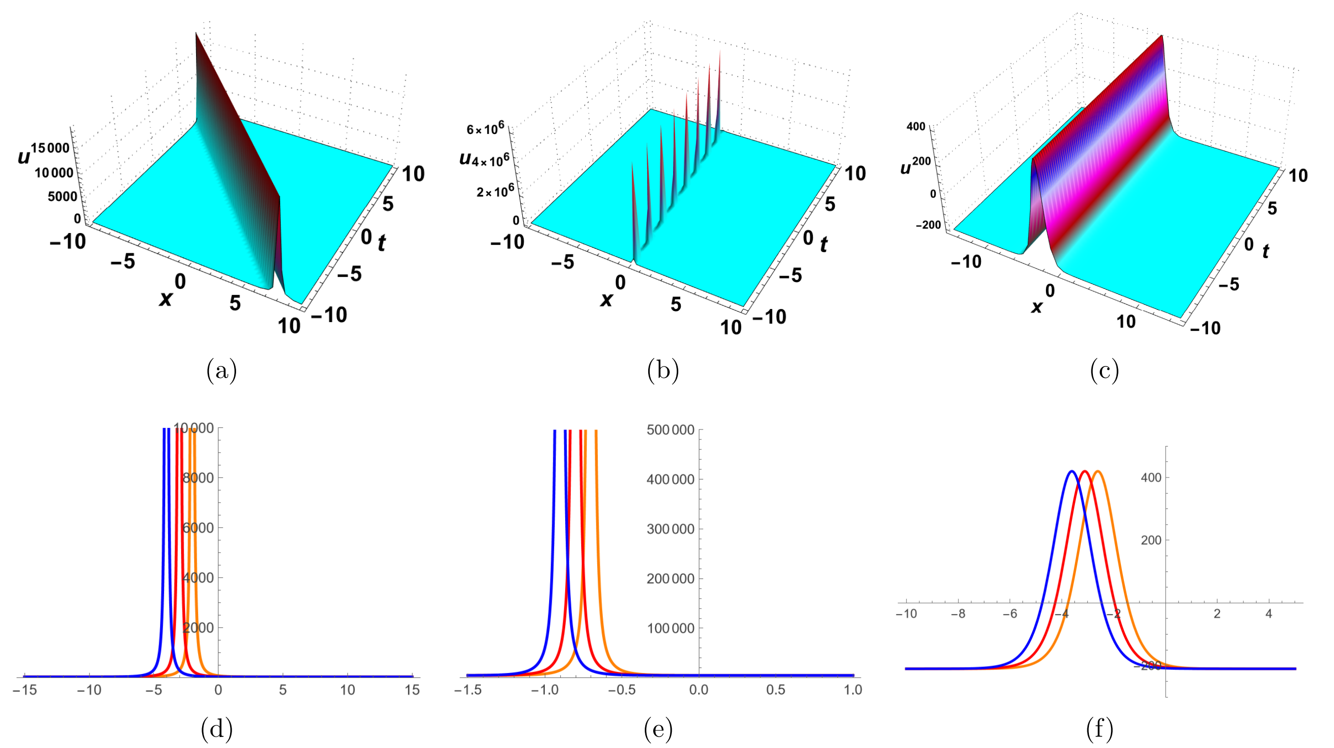

Figure 2: Graphics for the solutions (16)(a) and (b) for different parameter values and (17)(c) respectively, with their 2D plots in (d), (e), and (f), having parameters : (a) $j=1$, $s=1$, $\lambda=3$, $H_0=1$,$A=3$,$B=-330$,$C=90$; (b) $j=1$, $s=10$, $\lambda=8$, $H_0=1$,$A=5$,$B=-500$,$C=8$; (c) $t=0$, $j=1$, $s=2$, $\lambda=5$, $H_0=1$,$A=-4$,$B=400$,$C=10$

Family 3: For $[w_1,w_2,w_3,w_4]=[3,10,-2,-8]$ and $[\eta_1,\eta_2,\eta_3,\eta_4]=[2,1,1,2]$, then the equation (5) becomes,

$$M(\xi) = \frac{3e^{\xi}+10e^{2\xi}}{-8e^{\xi}-2e^{2\xi}} \tag{18}$$

Next, we substitute equation (18) into (8) and we get:

$$G(\xi) = H_0 + \frac{H_1(3e^{\xi} + 10e^{2\xi})}{(-8e^{\xi} - 2e^{2\xi})} + \frac{H_2(3e^{\xi} + 10e^{2\xi})^2}{(-8e^{\xi} - 2e^{2\xi})^2} + \frac{L_1(-8e^{\xi} - 2e^{2\xi})}{(3e^{\xi} + 10e^{2\xi})} + \frac{L_2(-8e^{\xi} - 2^{2\xi})^2}{(3e^{\xi} + 10e^{2\xi})^2} \tag{19}$$

Case 3.1:

$$H_1 = 0,\ H_2 = 0,\ j = \frac{1}{8}\left(\frac{Bs^3\sqrt{-\left(s^4\left(40A - B^2 - 4BC - 4C^2\right)\right)}}{A} - \frac{B^2s^5}{A} - \frac{2BCs^5}{A} + 12s^5\right)$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi) = \frac{2809}{5476}(\phi) + \frac{1935\left(-8e^{\xi} - 2e^{2\xi}\right)}{1369\left(3e^{\xi} + 10e^{2\xi}\right)}(\phi) + \frac{\left(-8e^{\xi} - 2e^{2\xi}\right)^2}{3748322A\left(3e^{\xi} + 10e^{2\xi}\right)^2}(\alpha)$$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(sx + jt + \lambda) = \frac{2809}{5476}(\phi) + \frac{1935\left(-8e^{sx+jt+\lambda} - 2e^{2(sx+jt+\lambda)}\right)}{1369\left(3e^{sx+jt+\lambda} + 10e^{2(sx+jt+\lambda)}\right)}(\phi) + \frac{\left(-8e^{sx+jt+\lambda} - 2e^{2(sx+jt+\lambda)}\right)^2}{3748322A\left(3e^{sx+jt+\lambda} + 10e^{2sx+jt+\lambda}\right)^2}(\alpha) \tag{20}$$

$$\phi = \left(\frac{\sqrt{-\left(s^4\left(40A - B^2 - 4BC - 4C^2\right)\right)}}{A} - \frac{Bs^2}{A} - \frac{2Cs^2}{A}\right)$$

$$\alpha = \left(-1848150Bs^2 - 3696300Cs^2 + \sqrt{\left(1848150Bs^2 + 3696300Cs^2\right)^2 - 136626336900000As^4}\right)$$

Case 3.2:

$$L_1 = 0,\ L_2 = 0,\ j = \frac{1}{8}\left(-\frac{Bs^3\sqrt{-\left(s^4\left(40A - B^2 - 4BC - 4C^2\right)\right)}}{A} - \frac{B^2s^5}{A} - \frac{2BCs^5}{A} + 12s^5\right)$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi) = \frac{2809}{5476}(\phi) + \frac{1032\left(3e^{\xi} + 10e^{2\xi}\right)}{1369\left(-8e^{\xi} - 2e^{2\xi}\right)}(\phi) + \frac{\left(3e^{\xi} + 10e^{2\xi}\right)^2}{3748322A\left(-8e^{\xi} - 2e^{2\xi}\right)^2}(\alpha)$$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(sx + jy + \lambda) = \frac{2809}{5476}(\phi) + \frac{1032\left(3e^{sx+jy+\lambda} + 10e^{2(sx+jy+\lambda)}\right)}{1369\left(-8e^{sx+jy+\lambda} - 2e^{2(sx+jy+\lambda)}\right)}(\phi) + \frac{\left(3e^{sx+jy+\lambda} + 10e^{2(sx+jy+\lambda)}\right)^2}{3748322A\left(-8e^{sx+jy+\lambda} - 2e^{2(sx+jy+\lambda)}\right)^2}(\alpha) \tag{21}$$

where

$$\phi = \left(-\frac{\sqrt{-\left(s^4\left(40A - B^2 - 4BC - 4C^2\right)\right)}}{A} - \frac{Bs^2}{A} - \frac{2Cs^2}{A}\right)$$

$$\alpha = \left(-525696Bs^2 - 1051392Cs^2 - \sqrt{\left(525696Bs^2 + 1051392Cs^2\right)^2 - 11054251376640As^4}\right)$$

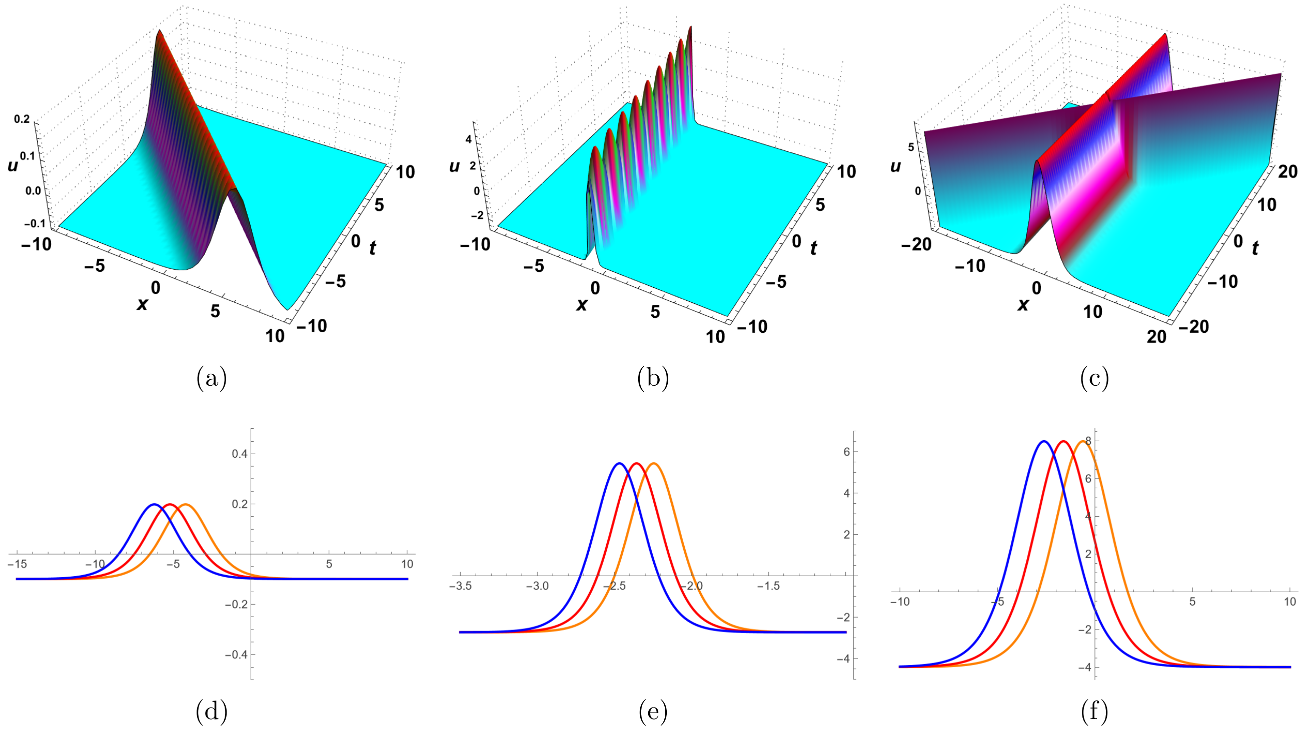

Figure 3: Graphics for the solutions (20) (a) and (b) for different parameter values and (21)(c) respectively, with their 2D plots in (d), (e), and (f), having parameters : (a) $j=1$, $s=1$, $\lambda=3$, $H_0=1$,$A=3$,$B=33$,$C=9$; (b) $j=1$, $s=9$, $\lambda=19$, $H_0=1$,$A=33$,$B=93$,$C=29$; (c) $t=0$, $j=1$, $s=1$, $\lambda=2$, $H_0=1$,$A=16$,$B=59$,$C=35$;

Family 4: For $[w_1,w_2,w_3,w_4]=[4,-2,4,2]$ and $[\eta_1,\eta_2,\eta_3,\eta_4]=[2,1,2,1]$, then equation (5) becomes,

$$M(\xi)=\frac{-2e^{\xi}+4e^{2\xi}}{2e^{\xi}+4e^{2\xi}} \tag{22}$$

Next, we substitute equation (22) into (8) and we get:

$$G(\xi)=H_0+\frac{H_1(-2e^{\xi}+4e^{2\xi})}{(2e^{\xi}+4e^{2\xi})}+\frac{H_2(-2e^{2\xi}+4e^{2\xi})^2}{(2e^{\xi}+4e^{2\xi})^2}+\frac{L_1(2e^{\xi}+4e^{2\xi})}{(-2e^{\xi}+4e^{2\xi})}+\frac{L_2(2e^{\xi}+4^{2\xi})^2}{(-2e^{\xi}+4e^{2\xi})^2} \tag{23}$$

Case 4.1:

$$L_1=0,\ L_2=0,H_1=0,\ j=\frac{1}{8}\left(\frac{Bs^3\sqrt{-\left(s^4\left(40A-B^2-4Bc-4c^2\right)\right)}}{A}-\frac{B^2s^5}{A}-\frac{2Bcs^5}{A}+12s^5\right)$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi)=\frac{1}{2}(\pi)+\frac{\left(4e^{2\xi}-2e^{\xi}\right)^2\left(-3Bs^2-6cs^2+\sqrt{\left(3Bs^2+6cs^2\right)^2-360As^4}\right)}{4A\left(2e^{\xi}+4e^{2\xi}\right)^2}$$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(jt+\lambda+sx)=\frac{1}{2}(\pi)+\frac{\left(-3Bs^2-6cs^2+\sqrt{\left(3Bs^2+6cs^2\right)^2-360As^4}\right)\left(4e^{2(jt+\lambda+sx)}-2e^{jt+\lambda+sx}\right)^2}{4A\left(2e^{jt+\lambda+sx}+4e^{2(jt+\lambda+sx)}\right)^2} \tag{24}$$

$$\pi=\left(-\frac{\sqrt{-\left(s^4\left(40A-B^2-4Bc-4c^2\right)\right)}}{A}+\frac{Bs^2}{A}+\frac{2cs^2}{A}\right)$$

Case 4.2:

$$L_1=0,\ L_2=0,H_1=0,\ j=\frac{1}{8}\left(-\frac{Bs^3\sqrt{-\left(s^4\left(40A-B^2-4Bc-4c^2\right)\right)}}{A}-\frac{B^2s^5}{A}-\frac{2Bcs^5}{A}+12s^5\right)$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi)=\tfrac{1}{2}(\pi)+\frac{\left(4e^{2\xi}-2e^{\xi}\right)^2\left(-3Bs^2-6cs^2-\sqrt{\left(3Bs^2+6cs^2\right)^2-360As^4}\right)}{4A\left(2e^{\xi}+4e^{2\xi}\right)^2}$$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(jt+\lambda+sx)=\frac{1}{2}(\pi)+\frac{\left(-3Bs^2-6cs^2-\sqrt{\left(3Bs^2+6cs^2\right)^2-360As^4}\right)\left(4e^{2(jt+\lambda+sx)}-2e^{jt+\lambda+sx}\right)^2}{4A\left(2e^{jt+\lambda+sx}+4e^{2(jt+\lambda+sx)}\right)^2} \tag{25}$$

where

$$\pi=\left(\frac{\sqrt{-\left(s^4\left(40A-B^2-4Bc-4c^2\right)\right)}}{A}+\frac{Bs^2}{A}+\frac{2cs^2}{A}\right)$$

Family 5: For $[w_1,w_2,w_3,w_4]=[6,7,5,-3]$ and $[\eta_1,\eta_2,\eta_3,\eta_4]=[2,1,1,2]$, then equation (5) becomes,

$$M(\xi)=\frac{7e^{\xi}+6e^{2\xi}}{5e^{\xi}-3e^{2\xi}} \tag{26}$$

Next, we substitute equation (30) into (8) and we get:

$$G(\xi)=H_0+\frac{H_1(7e^{\xi}+6e^{2\xi})}{(5e^{\xi}-3e^{2\xi})}+\frac{H_2(7e^{\xi}+6e^{2\xi})^2}{(5e^{\xi}-3e^{2\xi})^2}+\frac{L_1(5e^{\xi}-3e^{2\xi})}{(7e^{\xi}+6e^{2\xi})}+\frac{L_2(5e^{\xi}-3e^{2\xi})^2}{(7e^{\xi}+6e^{2\xi})^2} \tag{27}$$

Case 5.1:

$$H_1=0,\ H_2=0,\ j=\frac{1}{8}\left(\frac{Bs^3\sqrt{-\left(s^4\left(40A-B^2-4BC-4C^2\right)\right)}}{A}-\frac{B^2s^5}{A}-\frac{2BCs^5}{A}+12s^5\right)$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi)=\frac{551}{1156}(\phi)+\frac{126\left(5e^{\xi}-3e^{2\xi}\right)}{289\left(7e^{\xi}+6e^{2\xi}\right)}(\phi)+\frac{\left(5e^{\xi}-3e^{2\xi}\right)^2}{167042A\left(7e^{\xi}+6e^{2\xi}\right)^2}(\alpha)$$

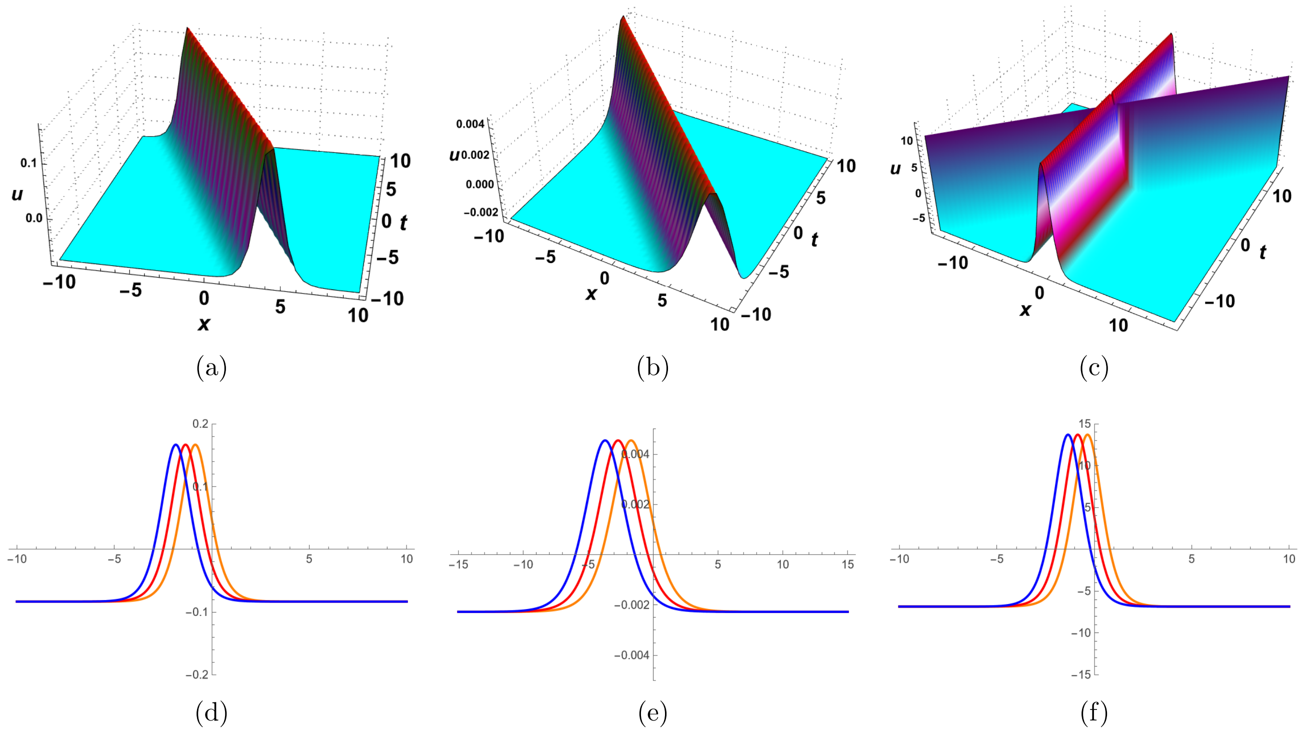

Figure 4: Graphics for the solutions (24) (a) and (b) for different parameter values and (25)(c) respectively, with their 2D plots in (d), (e), and (f), having parameters : (a) $j=1$, $s=2$, $\lambda=1$, $H_0=1$,$A=5$,$B=100$,$C=70$; (b) $j=1$, $s=1$, $\lambda=1$, $H_0=1$,$A=300$,$B=1200$,$C=500$; (c) $j=1$, $s=2$, $\lambda=0.001$, $H_0=1$,$A=40$,$B=70$,$C=35$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(sx+jt+\lambda)=\frac{551}{1156}(\phi)+\frac{126\left(5e^{sx+jt+\lambda}-3e^{2(sx+jt+\lambda)}\right)}{289\left(7e^{sx+jt+\lambda}+6e^{2(sx+jt+\lambda)}\right)}(\phi)+\frac{\left(5e^{(sx+jt+\lambda)}-3e^{2(sx+jt+\lambda)}\right)^2}{167042A\left(7e^{2(sx+jt+\lambda)}+6e^{2(sx+jt+\lambda)}\right)^2}(\alpha) \tag{28}$$

where,

$$\phi=\left(-\frac{\sqrt{-\left(s^4\left(40A-B^2-4BC-4C^2\right)\right)}}{A}+\frac{Bs^2}{A}+\frac{2Cs^2}{A}\right)$$

$$\alpha=\left(-339864Bs^2-679728Cs^2+\sqrt{\left(339864Bs^2+679728Cs^2\right)^2-4620301539840As^4}\right)$$

Case 5.2:

$$L_1=0,\ L_2=0,\ j=\frac{1}{8}\left(-\frac{Bs^3\sqrt{-\left(s^4\left(40A-B^2-4BC-4C^2\right)\right)}}{A}-\frac{B^2s^5}{A}-\frac{2BCs^5}{A}+12s^5\right)$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi)=\frac{551}{1156}(\phi)+\frac{45\left(7e^{\xi}+6e^{2\xi}\right)}{289\left(5e^{\xi}-3e^{2\xi}\right)}(\pi)+\frac{\left(7e^{\xi}+6e^{2\xi}\right)^2}{167042A\left(5e^{\xi}-3e^{2\xi}\right)^2}(\alpha)$$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(sx+jt+\lambda)=\frac{551}{1156}(\phi)+\frac{45\left(7e^{sx+jt+\lambda}+6e^{2(sx+jt+\lambda)}\right)}{289\left(5e^{sx+jt+\lambda}-3e^{2(sx+jt+\lambda)}\right)}(\pi)+\frac{\left(7e^{sx+jt+\lambda}+6e^{2(sx+jt+\lambda)}\right)^2}{167042A\left(5e^{(sx+jt+\lambda)}-3e^{2(sx+jt+\lambda)}\right)^2}(\alpha) \tag{29}$$

where,

$$\phi=\left(\frac{\sqrt{-\left(s^4\left(40A-B^2-4BC-4C^2\right)\right)}}{A}+\frac{Bs^2}{A}+\frac{2Cs^2}{A}\right)$$

$$\pi=\left(-\frac{\sqrt{-\left(s^4\left(40A-B^2-4BC-4C^2\right)\right)}}{A}-\frac{Bs^2}{A}-\frac{2Cs^2}{A}\right)$$

$$\alpha=\left(-43350Bs^2-86700Cs^2-\sqrt{\left(43350Bs^2+86700Cs^2\right)^2-75168900000As^4}\right)$$

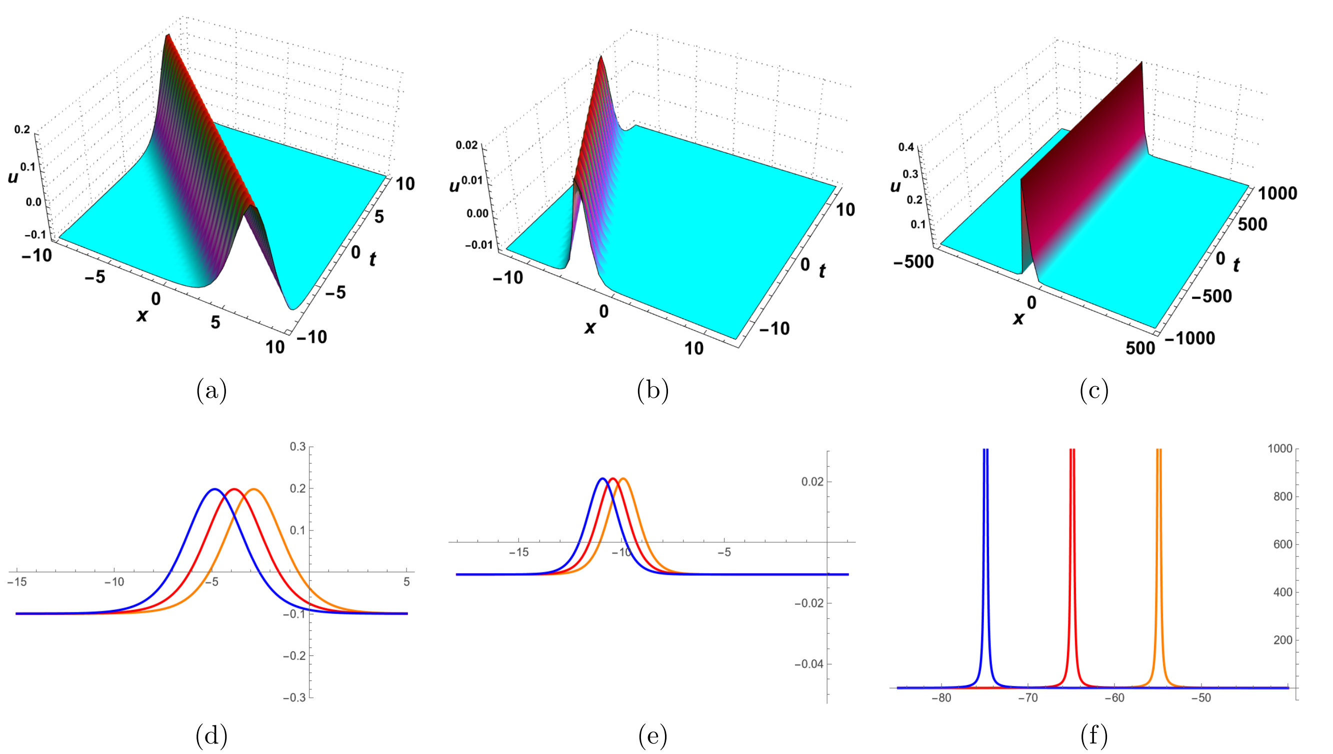

Figure 5: Graphics for the solutions (28) (a) and (b) for different parameter values and (29)(c) respectively, with their 2D plots in (d), (e), and (f), having parameters : (a) $j=1$, $s=1$, $\lambda=3$, $H_0=1$,$A=3$,$B=33$,$C=9$; (b) $j=1$, $s=2$, $\lambda=20$, $H_0=1$,$A=100$,$B=900$,$C=500$; (c) $j=1$, $s=0.1$, $\lambda=6$, $H_0=1$,$A=-500$,$B=1300$,$C=800$

Family 6: For $[w_1,w_2,w_3,w_4]=[-9,11,7,5]$ and $[\eta_1,\eta_2,\eta_3,\eta_4]=[3,2,2,3]$, then equation (5) becomes,

$$M(\phi)=\frac{11e^{2\phi}-9e^{3\phi}}{7e^{2\phi}+5e^{3\phi}} \tag{30}$$

Next, we substitute equation (30) into (8) and we get:

$$G(\xi)=H_0+\frac{H_1(11e^{2\xi}-9e^{3\xi})}{(7e^{2\xi}+5e^{3\xi})}+\frac{H_2(11e^{2\xi}-9e^{3\xi})^2}{(7e^{2\xi}+5e^{3\xi})^2}+\frac{L_1(7e^{2\xi}+5e^{3\xi})}{(11e^{2\xi}-9e^{3\xi})}+\frac{L_2(7e^{2\xi}+5e^{3\xi})^2}{(11e^{2\xi}-9e^{3\xi})^2} \tag{31}$$

Case 6.1:

$$H_1=0,\ H_2=0,\ j=\frac{1}{8}\left(\frac{Bs^3\sqrt{-\left(s^4\left(40A-B^2-4BC-4C^2\right)\right)}}{A}-\frac{B^2s^5}{A}-\frac{2BCs^5}{A}+12s^5\right)$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi)=\frac{3457}{6962}(\phi)+\frac{594\left(7e^{2\xi}+5e^{3\xi}\right)}{3481\left(11e^{2\xi}-9e^{3\xi}\right)}(\phi)+\frac{\left(7e^{2\xi}+5e^{3\xi}\right)^2}{48469444A\left(11e^{2\xi}-9e^{3\xi}\right)^2}(\alpha)$$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(M)=\frac{3457}{6962}(\phi)+\frac{594\left(7e^{2M}+5e^{3M}\right)}{3481\left(11e^{2M}-9e^{3M}\right)}(\phi)+\frac{\left(7e^{2M}+5e^{3M}\right)^2}{48469444A\left(11e^{2M}-9e^{3M}\right)^2}(\alpha) \tag{32}$$

where

$$M=sx+jt+\lambda$$

$$\phi=\left(-\frac{\sqrt{-\left(s^4\left(40A-B^2-4BC-4C^2\right)\right)}}{A}+\frac{Bs^2}{A}+\frac{2Cs^2}{A}\right)$$

$$\alpha=\left(-102351843Bs^2-204703686Cs^2+\sqrt{(102351843Bs^2+204703686Cs^2)^2-419035990619865960As^4}\right)$$

Case 6.2:

$$L_1=0,\ L_2=0,\ j=\frac{1}{8}\left(-\frac{Bs^3\sqrt{-\left(s^4\left(40A-B^2-4BC-4C^2\right)\right)}}{A}-\frac{B^2s^5}{A}-\frac{2BCs^5}{A}+12s^5\right)$$

Substituting above values of constants into the equation (10), we get a solution as:

$$G(\xi)=\frac{3457}{6962}(\phi)+\frac{210\left(11e^{2\xi}-9e^{3\xi}\right)}{3481\left(7e^{2\xi}+5e^{3\xi}\right)}(\pi)+\frac{\left(11e^{2\xi}-9e^{3\xi}\right)^2}{48469444A\left(7e^{2\xi}+5e^{3\xi}\right)^2}(\alpha)$$

Consequently, the exact solution for investigating equation(1) is obtained in the form :

$$G(sx+jt+\lambda)=\frac{3457}{6962}(\phi)+\frac{210\left(11e^{2(sx+jt+\lambda)}-9e^{3(sx+jt+\lambda)}\right)}{3481\left(7e^{2(sx+jt+\lambda)}+5e^{3(sx+jt+\lambda)}\right)}(\pi)+\frac{\left(11e^{2(sx+jt+\lambda)}-9e^{3(sx+jt+\lambda)}\right)^2}{48469444A\left(7e^{2(sx+jt+\lambda)}+5e^{3(sx+jt+\lambda)}\right)^2}(\alpha) \tag{33}$$

where,

$$\phi=\left(\frac{\sqrt{-\left(s^4\left(40A-B^2-4BC-4C^2\right)\right)}}{A}+\frac{Bs^2}{A}+\frac{2Cs^2}{A}\right)$$

$$\pi=\left(-\frac{\sqrt{-\left(s^4\left(40A-B^2-4BC-4C^2\right)\right)}}{A}-\frac{Bs^2}{A}-\frac{2Cs^2}{A}\right)$$

$$\alpha=\left(-12792675Bs^2-25585350Cs^2-\sqrt{(12792675Bs^2+25585350Cs^2)^2-6546101346225000As^4}\right)$$

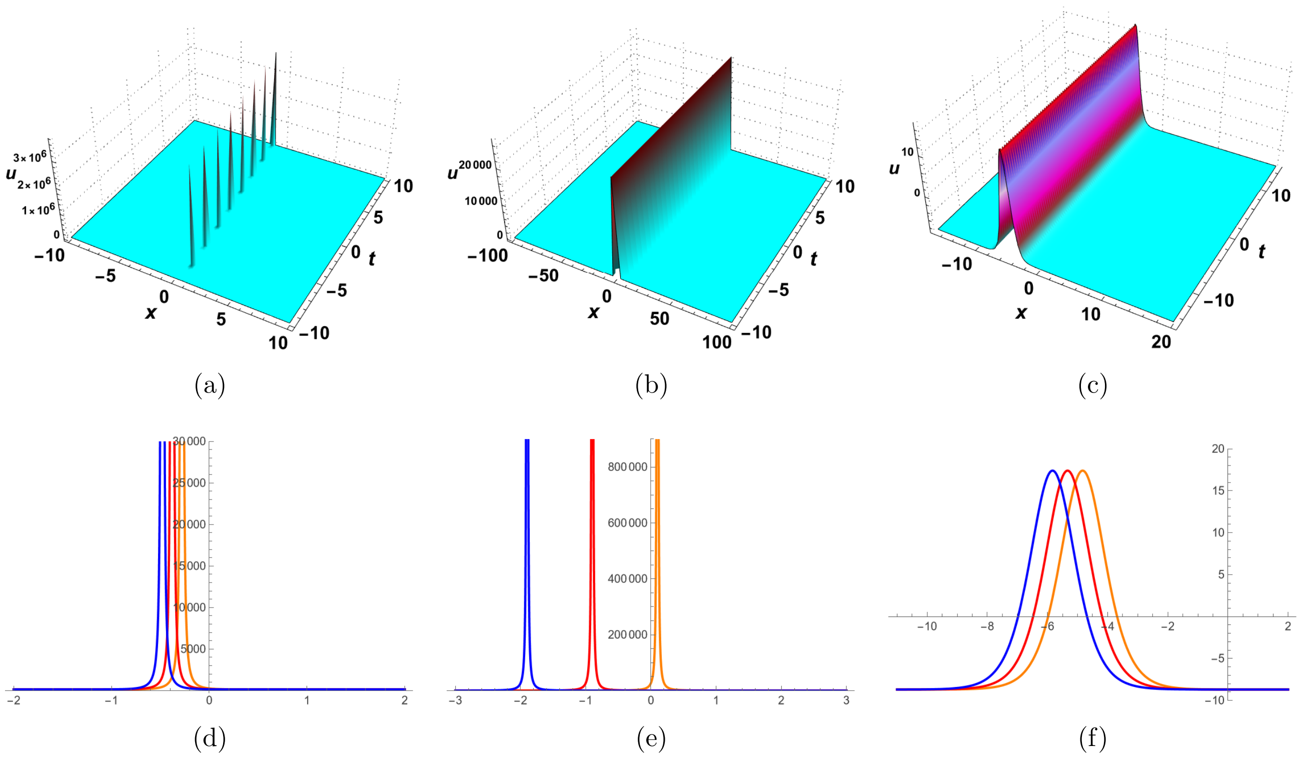

Figure 6: Graphics for the solutions (32) (a) and (b) for different parameter values and (33)(c), respectively, with their 2D plots in (d), (e), and (f), having parameters : (a) $j = 1$, $s = 10$, $\lambda = 3$, $H_0 = 1$,$A = 3$,$B = -30$,$C = 9$; (b) $j = 1$, $s = 1$, $\lambda = 0.1$, $H_0 = 1$,$A = 7$,$B = -400$,$C = 38$; (c) $j = 1$, $s = 2$, $\lambda = 10$, $H_0 = 1$,$A = 2$,$B = 5$,$C = 3$;

4 Results and Discussions

This work analysed different analytical solutions by considering the different families for the trial function in the utilized GERF method. This approach considers the trial function as a rational function with arbitrary parameters. For different values of arbitrary parameters, we obtain different trial functions and hence different analytical solutions. The dynamical behavior of these obtained solutions is analysed with the symbolic software *Mathematica,* for appropriate choices of the arbitrary parameters. The two-dimensional graphics are plotted for different time t values.

The starting choices of arbitrary parameters in the trial function give the opportunity to explore various analytical solutions that showcase diverse wave structures such as bright solitons, dark solitons, kink-type solitons, breathers, and periodic background waves. The studied solutions represent different types of wave structures depending on the arbitrary choice of parameters and exhibit wave behavior arising due to the interplay of nonlinearity and dispersion embedded in the investigated generalized fifth-order KdV-type equation.

The analysis for the drawn figures is as follows:

- In **Figure 1** Family 01 shows the rich variety of soliton wave structures obtained from the single solution for different parameters. As we can show in graph (a) is representing a dark soliton with a descending ridge of wave profile. The graph is showing inverted spikes, resembling a depression wave.

This type of wave structure is commonly observed in internal waves beneath the ocean surface where density dips occur. The corresponding 2D (d), shows downward peaks that represent its structure at particular time instances. The three different curves plotted for $t = 0, 1, 2$ (in different colors such as orange, red, and blue) illustrate the temporal evolution of the wave. This graph explains that the wave profile propagates towards the positive x-axis while preserving its shape, which is useful for predicting underwater wave propagation towards coastal regions. in graph (b), it depicts a bright soliton with a sharp localized peak. In the wave structure there is a singular elevation which indicates energy concentration in very sharp region. The 2D plot (e) of same wave profile depicts upward peaks over time maintaining stable amplitude and propagation along the positive x-direction. Further, graph (c) depicts a double interacting bright solitons structure, where two soliton interacting with each other above surface bright soliton, wave profiles cross each othet in middle at a point. The corresponding 2D plot (f) depicts smooth symmetric peak that is propogating with x axis while maintaining is amplitude.

- **Figure 2** in Family 2 the rich diversity of soliton wave forms derived from the single solution for various parameters is displayed. As we can see in graph (a), it depicts a bright soliton, where we can obsereve a sharp localised peak. A single rise in the wave profile indicates that energy is concentrated in a narrow area. The corresponding 2D plot (d) shows upward peak that maintain its amplitude over time. The time evolution ($t = 0, 1, 2$) in graph confirms indicates that the structure propagates smoothly along the positive x-axis direction. Graph (b) depicts a periodic wave train with multiple peaks, forming a repeating pattern along the spatial domain. This indicates a cnoidal wave structure, The 2D plot (e) confirms the periodic oscillations, showing consistent amplitude and wavelength across time. Graph (c) shows a localized bright soliton with slight oscillatory background, indicating a modulated wave. The 2D plot (f) shows stable peak propagation in positive x direction, confirming its soliton behavior.

- **Figure 3:** Family 03 shows the rich variety of soliton wave structures obtained from the single solution for different parameters. As we can show in graph (a), it represents a bright soliton with smooth bell-shaped profile, This indicates stable energy localization. The corresponding 2D plot (d) shows a monotonic transition curve at different times, indicating that the wave propagates without changing its shape. This confirms its stability and is useful in studying switching phenomena in physical systems. Graph (b) depicts a periodic-soliton hybrid structure, where oscillations are superimposed on a localized wave. This type of structure is observed in nonlinear lattices and fluid wave interactions. The 2D plot (e) shows oscillatory peaks moving along the spatial direction. Graph (c) shows bright soliton type structure with smooth bell type structure, double spikes interacting each other. The 2D plot (f) confirms periodic oscillation in amplitude with time.

- In **Figure 4**, Family 04 shows the rich variety of soliton wave structures obtained from the single solution for different parameters. As we can show in graph (a), it represents a bright soliton with smooth bell-shaped profile. This indicates stable energy localization.The corresponding 2D plot (d) shows a symmetric peak that maintains its amplitude over time, confirming non-dispersive propagation along the positive x-axis. Graph (b) shows a similar garphical structre as (a) as they both plotted from same solution for diffrent parameters this graph is depitcting a bright soliton with smooth bell-shaped profile. The corresponding 2D plot (e) depicts Non-dispersive propagation along the positive x-axis is confirmed by a symmetric peak that keeps its amplitude over time. Graph (c) depicts a double solitons structure, where two soliton interacting with each other above surface bright soliton, wave profiles cross each othet in middle at a point. The corresponding 2D plot (f) depicts smooth symmetric peak that is propogating with x axis while maintaining is amplitude. confirms alternating peaks and dips, showing stable propagation behavior.

- **Figure 5** Family 05 shows the soliton wave structures obtained from the single solution for different parameters. As we can show in graph (a), it represents a bright soliton with smooth bell-shaped profile. The corresponding 2D plot (d)shows a symmetric peak that maintains its amplitude over time, confirming non-dispersive propagation along the positive x-axis . Graph (b) depicts a high-amplitude singular wave, where peaks become extremely sharp and intense. This type of structure is useful in studying extreme wave events and nonlinear instabilities. The 2D plot (e) confirms sharp peaks at different time levels. Graph (c) shows a singular localized soliton, indicating stable wave motion with constant frequency. The 2D plot (f) confirms sinusoidal like propagation, maintaining amplitude value over time.

- In **Figure 6**, Family 06 shows the rich variety of soliton wave structures obtained from the single solution for different parameters. As we can show in graph (a), it represents a single train of spike solitons, forming a periodic sequence of sharp peaks along the spatial domain. Forming a periodic sequence of sharp peaks. This structure observed in pulse trains. The corresponding 2D plot (d) shows repeated spikes across staining unifpace, mainorm amplitude over time, indicating coherent propagation. Graph (b) depicts a singular spike bright soliton, singular spike in wave profile indicating eneregy concentration in localised area with strong nonlinearity and dispersion effects. The 2D plot (e) shows stable amplitude and propogation with time. Graph (c) represents a singular localized soliton , indicating a strong energy concentration. The 2D plot (f) confirms sharp peak propagation and stability along the spatial direction.

5 Conclusions

In this work, the generalized exponential rational function method (GERFM) was successfully applied to investigate the analytical solutions of a generalized nonlinear fifth-order KdV-type partial differential equation, which contains the Sawada–Kotera equation as its particular case. Then we employed a traveling wave transformation on the governing PDE. On this reduced ordinary differential equation the homogeneous balancing principle was used to evaluate the truncation order of the trial solution. A rich set of exact analytical solutions in closed forms was obtained by taking into account six parameter families of the rational function $M(\xi)$.

The wide range of nonlinear wave structures such as bright solitons, dark solitons, kink-type solitons, dark-bright composite solitons, breather-type oscillations, periodic multi-peak wave trains and large-amplitude singular waves are displayed by the derived solutions. Three- and two-dimensional time-evolution profiles were created in *Mathematica*. These graphics were analysed for dynamical behavior of the solutions. The analysis confirmed the shape-preserving and propagate stably for the solutions with balance between the nonlinearity and the higher-order dispersion terms. The wave propagation was further determined by 2D plots at time values $(0, 1, 2)$. And, these graphics validated the soliton behaviour of the derived solutions.

We demonstrated GERFM as a powerful, transparent, and broadly applicable approach to generate exact solutions. This method does not require the Painlevé integrability. The investigated KdV-type equation has direct physical relevance in numerous fields of science and engineering. The results of this work contribute to soliton theory and provide a foundation for further analytical investigations of this equation.

5.1 Future scope of the work

The GERF method can be utilized to investigate more complex nonlinear PDEs in higher-dimensional systems, variable-coefficient equations, and coupled nonlinear models. This technique can construct different novel solution classes such as interaction solutions, hybrid soliton structures, and multi-wave configurations.

With nonlocal operators and fractional-order derivatives into the current framework could provide profound understanding of memory-dependent systems and anomalous dispersion. Stability analysis and perturbation studies can be extended to derive solutions to physical robustness. The integration of symbolic and numerical techniques could enhance the automation and scalability of the GERFM for complex systems.

The results in this research provide have a solid foundation for advancing analytical solution and methods. It contributes to the broader development of nonlinear wave theory having potential applications among applied mathematics, physics, and engineering disciplines.

Declarations

Ethical declaration

Not applicable.

Competing interests

There is no conflict of interest, according to the author and the supervisor.

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